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Education for All

Dr. Babasaheb Ambedkar
Open University
(Established by Government of Gujarat)

NAAC ACCREDITED A++ GRADE 3.55/4

BBA/DBA

SEMESTER - 1

BBAMDC103

DBAMDC103

Business Mathematics

Message for the Students

Dr. Babasaheb Ambedkar Open (University is the only state Open University, established by the Government of Gujarat by the Act No. 14 of 1994 passed by the Gujarat State Legislature; in the memory of the creator of Indian Constitution and Bharat Ratna Dr. Babasaheb Ambedkar. We Stand at the seventh position in terms of establishment of the Open Universities in the country. The University provides as many as 54 courses including various Certificate, Diploma, UG, PG as well as Doctoral to strengthen Higher Education across the state.

On the occasion of the birth anniversary of Babasaheb Ambedkar, the Gujarat government secured a quiet place with the latest convenience for University, and created a building with all the modern amenities named 'Jyotirmay' Parisar. The Board of Management of the University has greatly contributed to the making of the University and will continue to this by all the means.

Education is the perceived capital investment. Education can contribute more to improving the quality of the people. Here I remember the educational philosophy laid down by Shri Swami Vivekananda:

> ***"We want the education by which the character is formed, strength of mind is Increased, the intellect is expand and by which one can stand on one's own feet".***

In order to provide students with qualitative, skill and life oriented education at their threshold. Dr. Babaasaheb Ambedkar Open University is dedicated to this very manifestation of education. The university is incessantly working to provide higher education to the wider mass across the state of Gujarat and prepare them to face day to day challenges and lead their lives with all the capacity for the upliftment of the society in general and the nation in particular.

The university following the core motto 'स्वाध्यायः परमम् तपः' does believe in offering enriched curriculum to the student. The university has come up with lucid material for the better understanding of the students in their concerned subject. With this, the university has widened scope for those students who are not able to continue with their education in regular/conventional mode. In every subject a dedicated term for Self Learning Material comprising of Programme advisory committee members, content writers and content and language reviewers has been formed to cater the needs of the students.

Matching with the pace of the digital world, the university has its own digital platform Omkar-e to provide education through ICT. Very soon, the University going to offer new online Certificate and Diploma programme on various subjects like Yoga, Naturopathy, and Indian Classical Dance etc. would be available as elective also.

With all these efforts, Dr. Babasaheb Ambedkar Open University is in the process of being core centre of Knowledge and Education and we invite you to join hands to this pious *Yajna* and bring the dreams of Dr. Babasaheb Ambedkar of Harmonious Society come true.

Prof. Ami Upadhyay
Vice Chancellor,
Dr. Babasaheb Ambedkar Open University,
Ahmedabad.

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BBAMDC103

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BUSINESS MATHEMATICS

BLOCK-1

BLOCK-2

BLOCK-3

PART - 1

BBA

SEMESTER-1

BUSINESS MATHEMATICS

BLOCK: 1

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Publisher's Name: Dr. Ajaysinh Jadeja
Registrar,
Dr. Babasaheb Ambedkar Open University
'JyotirmayParisar',Opp.ShriBalaji
Temple,Chharodi,Ahmedabad,382481,
Gujarat,India.

Edition: 2024 (First Edition)

ISBN: **978-93-5598-883-6**

978-93-5598-883-6

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UNIT-1	INDICES AND SURDS

1.1 Indices

1.2 Positive indices

1.3 Fractional index

1.4 Surds

1.5 Operations with surds

1.6 Rationalising factor

1.1 Indices

We are aware of certain operations of addition and multiplication and now we take up certain higher order operations with powers and roots under the respective heads of indices and surds.

The knowledge of these rules is indispensable for any serious mathematical manipulation. We will deal with indices and surds in this chapter.

1.2 Positive Indices

In a positive index the base multiplies a given number of times depending on the power or the value of the index. In case of a negative index, it is reciprocal of the base which multiplies a number of times depending on the value of the negative index. The formal definition and the fundamental rules of operations with positive index are given below which would be relevant in other cases also.

Definition : If n is a positive integer, and 'x' is i)a real number, i.e., $n \in N$ and $x \in R$, x is used to denote the continued product of n factors each equal to x shown below:

$$x^n = x \times x \times \ldots\ldots \text{ to } n \text{ factors.}$$

where ***n*** is called the index or the exponent of base x.

[1] Multiplication of Factors with Same Bases

$$x^m \times x^n = x^{m+n}$$

Proof : $x^m \times x^n = (x \cdot x \cdot x \cdot \ldots\ldots to\ m \text{ factors}) \times (x \cdot x \cdot x \cdot \ldots\ldots to\ n \text{ factors})$

$$= x \cdot x \cdot x \cdot \ldots\ldots \text{to } (m+n) \text{factors}$$

$$= x^{m+n} \text{ (by definition)}$$

This is called the fundamental index Law.

Ex.1 $x^{\frac{3}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$ can also be expressed as $x^{\frac{3}{2}+\frac{1}{2}+\frac{1}{2}}$ or $x^{\frac{5}{2}}$

[2] Multiplication of Factors with Different Bases

The rules for this can be stated as follows. :

(i) $x^m \times y^m = (xy)^m$ (ii) $x^m \cdot y^m \cdot z^m = (xyz)^m$

$\therefore 4^4 \times 7^4 = (4 \times 7)^4 = (28)^4$

[3] Division of Factors with Same Bases

$x^m \div x^n = x^{m-n}$

Proof : $\dfrac{x^m}{x^n} = \dfrac{x \cdot x \cdot x \cdot .....\text{to } m \text{ factors}}{x \cdot x \cdot x \cdot ....\text{to } n \text{ factors}}$

(i) If $m > n$, there will be $(m - n)$ factors 'x', it being more in the numerator than in the denominator.

$x^m \div x^n = x \cdot x \cdot x \cdot ....\text{to} (m-n)$ factors $= x^{m-n}$

(ii) If $m = n$, there are same number of factors of x in the numerator and denominator which cancel away.

(iii) If $m < n$, there are $(n - m)$ extra factors of x in the denominator.

$$x^m \div x^n = \frac{1}{x \cdot x \cdot x \cdot .....\text{to } (n-m) \text{factors}} = \frac{1}{x^{n-m}}$$

(iv) $\dfrac{1}{x^{m-n}} = \dfrac{1}{x^m \times x^{-n}} = x^{-m} \times x^n = x^{n-m}$

(v) $x^{-m} = \dfrac{1}{x^m}$; or $\dfrac{1}{x^m} = x^{-m}$ where $x \neq 0$ or 1

[4] Division of Factors with Different Bases

The rule for the purpose can be stated as follows. :

$$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$$

Proof : $\left(\dfrac{x}{y}\right)^m = \left(\dfrac{x}{y}\right) \cdot \left(\dfrac{x}{y}\right) \cdot \left(\dfrac{x}{y}\right).....\text{to } m$ factors

$$= \frac{x \cdot x \cdot x......\text{to } m \text{ factors}}{y \cdot y \cdot y......\text{to } m \text{ factors}} = \frac{x^m}{y^m}$$

There can be negative integral index to any base except 0 and 1 in a power function. When negative integral index is there, the power function becomes the reciprocal of the function having a positive index.

$$\frac{x^{-m}}{y^{-n}} = \frac{\frac{1}{x^m}}{\frac{1}{y^n}} = \frac{1}{x^m} \times \frac{y^n}{1} = \frac{y^n}{x^m}$$

[5] Power Function

$$\left(x^m\right)^n = x^{mn}$$

For Example: (i) $\left(x^3\right)^2 = x^6$ (ii) $\left(4x^2\right)^3 = 4^3 \cdot x^6 = 64x^6$

A power function can be raised to a power as given below:

$$x^{(m)^n} = x^{m^n}$$

For Example: (i) $a^{(3)^2} = a^9$ (ii) $3^{3^2} = 3^9$

[6] Zero and Unity Index

The general principle is that anything other than zero raised to the power zero is one, i.e., $x^0 = 6^0 = (-4)^0 = 1, \quad (x \neq 0)$

Thus, $x^0 \times x^n = x^{0+n} = x^n$ or $x^n \times x^{-n} = x^{n-n} = x^0 = 1$

As a rule, any base raised to unity or 1 is equal to the base itself.

$$x^1 = x; \; 4^1 = 4$$

Ex.2 Simplify $\frac{\left(4x^2\right)^3}{\left(2x^3\right)^2} + \frac{\left(6x^3\right)^2}{\left(3x^3\right)^2}$

Solution : $\frac{\left(4x^2\right)^3}{\left(2x^3\right)^2} + \frac{\left(6x^3\right)^2}{\left(3x^3\right)^2} = \frac{4^3x^6}{2^3x^6} + \frac{6^2x^6}{3^3x^6}$

$$= \frac{64x^6}{4x^6} + \frac{36x^6}{27x^6}$$

$$= 16x^{6-6} + \frac{4}{3}x^{6-6}$$

$$= 16x^0 + \frac{4}{3}x^0 \quad \left(\because x^0 = 1\right)$$

$$= 16 + \frac{4}{3} = \frac{52}{3}$$

Ex.3 Simplify $\frac{3^5 \cdot 27^3 \cdot 9^4}{3 \cdot (81)^4}$

Solution : $\frac{3^5 \cdot 27^3 \cdot 9^4}{3 \cdot (81)^4} = \frac{3^5 \cdot \left(3^3\right)^3 \cdot \left(3^2\right)^4}{3 \cdot \left(3^4\right)^4}$

$$= \frac{3^5 \cdot 3^9 \cdot 3^8}{3 \cdot 3^{16}}$$

$$= \frac{3^{5+9+8}}{3^{1+16}}$$

$$= \frac{3^{22}}{3^{17}} = 3^{22-17} = 3^5$$

Ex.4 If $x^3 = 4$ and $y^2 = x$, find y^6 and y^{12}

Solution : If $y^2 = x$, then $y^6 = x^3 = 4$; and

$$y^{12} = \left(x^3\right)^2 = \left(4\right)^2 = 16$$

1.3 Fractional Index

In a positive fractional index, the numerator represents the power and the denominator represents the root.

Meaning of $x^{\frac{p}{q}}$, where *p* and *q* are any two positive integers.

Since $x^m \times x^n = x^{m+n}$ holds true for all values of *m* and *n*,

putting $m = n = \frac{p}{q}$, we have

$$x^{\frac{p}{q}} \times x^{\frac{p}{q}} = x^{\frac{p}{q}+\frac{p}{q}} = x^{2\frac{p}{q}}$$

Similarly, $x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times x^{\frac{p}{q}} \times \ldots\ldots \text{to } q \text{ factors} = x^{\left(\frac{p}{q}\right)\times q} = x^p$

$$\therefore\ x^{\frac{p}{q}} = \sqrt[q]{x^p}$$

Therefore, $x^{\frac{p}{q}}$, represents the q^{th} root of the p^{th} power of *x*.

Similarly, $x^{\frac{p}{q}} = \left(\sqrt[q]{x}\right)^p$, represent the p^{th} power of the q^{th} root of *x*.

In case, the fractional index is negative, the function is transformed into the reciprocal of one with a positive fractional index as shown below:

For Example: (i) $x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}}$ (ii) $x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}$

(iii) $8^{-\frac{2}{3}} = \frac{1}{8^{2/3}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$

(iv) $\sqrt[3]{x^5} = \left(\sqrt[3]{x}\right)^5 = x^{\frac{5}{3}} = \left(x^5\right)^{\frac{1}{3}} = \left(x^{\frac{1}{3}}\right)^5$

(v) $16^{-\frac{3}{4}} = \frac{1}{\sqrt[4]{16^3}} = \frac{1}{\sqrt[4]{4096}} = \frac{1}{8}$

(vi) $8^{-4/3} = \frac{1}{\sqrt[3]{8^4}} = \frac{1}{\sqrt[3]{4096}} = \frac{1}{16}$

Ex.5 To which power should we raise $x^{\frac{2}{3}}$ to get x?

Solution : $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = x^{\frac{2}{3}\times\frac{3}{2}} = x^1 = x$

Ex.6 Simplify $\frac{x^{\frac{4}{7}} \cdot \sqrt[5]{x^3} \cdot \sqrt[7]{x^3}}{\sqrt[5]{x^8}}$

Solution : Let $\frac{x^{\frac{4}{7}} \cdot \sqrt[5]{x^3} \cdot \sqrt[7]{x^3}}{\sqrt[5]{x^8}} = \frac{x^{\frac{4}{7}} \cdot x^{\frac{3}{5}} \cdot x^{\frac{3}{7}}}{x^{\frac{8}{5}}}$

$$= \frac{x^{\frac{4}{7}+\frac{3}{5}+\frac{3}{7}}}{x^{\frac{8}{5}}}$$

$$= \frac{x^{1+\frac{3}{5}}}{x^{\frac{8}{5}}} = \frac{x^{\frac{8}{5}}}{x^{\frac{8}{5}}} = x^{\frac{8}{5}-\frac{8}{5}} = x^0 = 1$$

Ex.7 Simplify (i) $\frac{2^5(2^3)^3 2^4}{2(2^4)^4}$ (ii) $\frac{9^{18}\sqrt{6561}}{27^3\, 3^{14}}$

Solution : (i) $\frac{2^5(2^3)^3 2^4}{2(2^4)^4} = \frac{2^5 2^9 2^4}{2\ 2^{16}}$

$$= \frac{2^{5+9+4}}{2^{1+16}}$$

$$= \frac{2^{18}}{2^{17}} = 2$$

(ii) $\frac{9^{18}\sqrt{6561}}{27^3\, 3^{14}} = \frac{(3^2)^{18}\sqrt{3^8}}{(3^3)^3\, 3^{14}}$

$$= \frac{(3^{36})(3^4)}{(3^9)\, 3^{14}}$$

$$= \frac{3^{36+4}}{3^{14+9}}$$

$$= \frac{3^{40}}{3^{23}} = 3^{40-23} = 3^{17}$$

Ex.8 Simplify (i) $\frac{(2^{2n} - 3\times 2^{2n-2})(3^n - 2\times 3^{n-2})}{3^{n-4}(4^{n+3} - 2^{2n})}$

(ii) $\frac{6^{n+1}\times 3^{2m-n}\times 5^{n+m+3}\times 2^{m+3}}{6^{m+1}\times 10^{n+3}\times 15^m}$

Solution : (i)

$$\frac{(2^{2n}-3\times 2^{2n-2})(3^n-2\times 3^{n-2})}{3^{n-4}(4^{n+3}-2^{2n})}=\frac{(2^{2n}-3\times 2^{2n}\times 2^{-2})(3^n-2\times 3^n\times 3^{-2})}{3^{n-4}((2^2)^{n+3}-2^{2n})}$$

$$=\frac{(2^{2n}-3\times 2^{2n}\times 2^{-2})(3^n-2\times 3^n\times 3^{-2})}{3^{n-4}(2^{2n+6}-2^{2n})}$$

$$=\frac{(2^{2n}-3\times 2^{2n}\times 2^{-2})(3^n-2\times 3^n\times 3^{-2})}{3^n\times 3^{-4}(2^{2n}\times 2^6-2^{2n})}$$

$$=\frac{2^{2n}(1-3\times 2^{-2})3^n(1-2\times 3^{-2})}{3^n\times 3^{-4}\times 2^{2n}(2^6-1)}$$

$$=\frac{(1-3\times 2^{-2})(1-2\times 3^{-2})}{3^{-4}\times(2^6-1)}$$

$$=\frac{(1-\frac{3}{4})(1-\frac{2}{9})}{\frac{1}{81}\times(64-1)}$$

$$=\frac{(\frac{1}{4}\times\frac{7}{9})}{\frac{1}{81}\times(63)}\qquad=\frac{1}{4}$$

Solution: (ii)

$$\frac{6^{n+1}\times 3^{2m-n}\times 5^{n+m+3}\times 2^{m+3}}{6^{m+1}\times 10^{n+3}\times 15^m}=\frac{(2\times 3)^{n+1}\times 3^{2m-n}\times 5^{n+m+3}\times 2^{m+3}}{(2\times 3)^{m+1}\times(5\times 2)^{n+3}\times(5\times 3)^m}$$

$$=\frac{2^{n+1}\times 3^{n+1}\times 3^{2m-n}\times 5^{n+m+3}\times 2^{m+3}}{2^{m+1}\times 3^{m+1}\times 5^{n+3}\times 2^{n+3}\times 5^m\times 3^m}$$

$$=\frac{2^{n+1+m+3}\times 3^{n+1+2m-n}\times 5^{n+m+3}}{2^{m+1+n+3}\times 3^{m+1+m}\times 5^{n+3+m}}$$

$$=\frac{2^{n+m+4}\times 3^{1+2m}\times 5^{n+m+3}}{2^{n+m+4}\times 3^{1+2m}\times 5^{n+m+3}}=1$$

Ex.9 Simplify $\left(\frac{x^c}{x^a}\right)^{c+a}\left(\frac{x^b}{x^c}\right)^{b+c}\left(\frac{x^a}{x^b}\right)^{a+b}$

Solution:

$$\left(\frac{x^c}{x^a}\right)^{c+a}\left(\frac{x^b}{x^c}\right)^{b+c}\left(\frac{x^a}{x^b}\right)^{a+b}=\left(x^{c-a}\right)^{c+a}\left(x^{b-c}\right)^{b+c}\left(x^{a-b}\right)^{a+b}$$

$$=x^{(c-a)(c+a)}x^{(b-c)(b+c)}x^{(a-b)(a+b)}$$

$$=x^{(c^2-a^2)}x^{(b^2-c^2)}x^{(a^2-b^2)}$$

$$=x^{c^2-a^2+b^2-c^2+a^2-b^2}$$

$$=x^0=1$$

Ex.10 Simplify $\frac{1}{1+x^{a-b}+x^{a-c}}+\frac{1}{1+x^{b-a}+x^{b-c}}+\frac{1}{1+x^{c-a}+x^{c-b}}$

Solution: We simplify given value term by term as under:

$$\frac{1}{1+x^{a-b}+x^{a-c}}=\frac{1}{x^{a-a}+x^{a-b}+x^{a-c}} \qquad (x^a x^{-a}=1)$$

$$=\frac{1}{x^a x^{-a}+x^a x^{-b}+x^a x^{-c}}$$

$$=\frac{1}{x^a(x^{-a}+x^{-b}+x^{-c})} \qquad \ldots\ldots \text{(i)}$$

$$\frac{1}{1+x^{b-a}+x^{b-c}}=\frac{1}{x^{b-b}+x^{b-a}+x^{b-c}} \qquad (x^b x^{-b}=1)$$

$$=\frac{1}{x^b x^{-b}+x^b x^{-a}+x^b x^{-c}}$$

$$=\frac{1}{x^b(x^{-b}+x^{-a}+x^{-c})} \qquad \ldots\ldots\text{(ii)}$$

$$\frac{1}{1+x^{c-a}+x^{c-b}}=\frac{1}{x^{c-c}+x^{c-a}+x^{c-b}} \qquad (x^c x^{-c}=1)$$

$$=\frac{1}{x^c x^{-c}+x^c x^{-a}+x^c x^{-b}}$$

$$=\frac{1}{x^c(x^{-c}+x^{-a}+x^{-b})} \qquad \ldots\ldots\text{(iii)}$$

By adding (i), (ii) and (iii), we have

$$\frac{1}{1+x^{a-b}+x^{a-c}}+\frac{1}{1+x^{b-a}+x^{b-c}}+\frac{1}{1+x^{c-a}+x^{c-b}}$$

$$=\frac{1}{x^a(x^{-a}+x^{-b}+x^{-c})}+\frac{1}{x^b(x^{-b}+x^{-a}+x^{-c})}+\frac{1}{x^c(x^{-c}+x^{-a}+x^{-b})}$$

$$=\frac{1}{(x^{-a}+x^{-b}+x^{-c})}\left[\frac{1}{x^a}+\frac{1}{x^b}+\frac{1}{x^c}\right]$$

$$=\frac{1}{(x^{-a}+x^{-b}+x^{-c})}(x^{-a}+x^{-b}+x^{-c})=1$$

Ex.11 If $2^3\times 3^4\times 72=6^x$, then find the value of x

Solution: $2^3\times 3^4\times 72=6^x$

$$2^3\times 3^4\times 2\times 36=6^x$$

$$2^3\times 3^4\times 2\times 6^2=6^x$$

$$2^4\times 3^4\times 6^2=6^x$$

$$(2\times 3)^4\times 6^2=6^x$$

$$6^4\times 6^2=6^x$$

$$6^6=6^x$$

$$x=6$$

Ex.12 If $x=(10+\sqrt{25})(12-\sqrt{49})$ then find the value of $\sqrt{x}$.

Solution: $x=(10+\sqrt{25})(12-\sqrt{49})$

$$x=(10+5)(12-7)$$

$$x=(15)(5)$$

$$x=75$$

$$\sqrt{x}=\sqrt{75}$$

$$\sqrt{x}=\sqrt{25\times 3}$$

$$\sqrt{x}=5\sqrt{3}$$

Ex.13 Simplify (i) $\dfrac{5^{x+2}-(25)5^{x-1}}{10\ \ 5^{x}}$ (ii) $\dfrac{(4^{x})^{2}9}{16^{x+1}-2^{x+1}8^{x}}$ (iii) $\dfrac{3^{x}+3^{x-1}}{3^{x+1}-3^{x}}$

Solution:

(i) $$\frac{5^{x+2}-(25)5^{x-1}}{10\ \ 5^{x}}=\frac{5^{x}5^{2}-(5^{2})5^{x}5^{-1}}{(10)\ 5^{x}}$$

$$=\frac{5^{x}5^{2}-(5^{2})5^{x}5^{-1}}{(10)\ \ 5^{x}}$$

$$=\frac{5^{x}(5^{2}-5^{2-1})}{(10)\ \ 5^{x}}$$

$$=\frac{(25-5)}{10}=2$$

(ii) $$\frac{(4^{x})^{2}9}{16^{x+1}-2^{x+1}8^{x}}=\frac{(2^{2x})^{2}3^{2}}{(2^{4})^{x+1}-2^{x+1}2^{3x}}$$

$$=\frac{2^{4x}\ 3^{2}}{2^{4x+4}-2^{x+1+3x}}$$

$$=\frac{2^{4x}\ 3^{2}}{2^{4x+4}-2^{4x+1}}$$

$$=\frac{2^{4x}\ 3^{2}}{2^{4x}(2^{4}-2^{1})}$$

$$=\frac{3^{2}}{(16-2)}\ =\frac{9}{14}$$

(iii) $$\frac{3^{x}+3^{x-1}}{3^{x+1}-3^{x}}=\frac{3^{x}+3^{x}\ 3^{-1}}{3^{x}\ 3^{1}-3^{x}}$$

$$=\frac{3^{x}(1+3^{-1})}{3^{x}\ (3-1)}$$

$$=\frac{1+3^{-1}}{3-1}$$

$$=\frac{1+1/3}{2}=\frac{4/3}{2}=\frac{2}{3}$$

Ex.14 Find the value of *x* if $2^{x-1}+2^{x+1}=320$.

Solution: $2^{x-1}+2^{x+1}=320$

$$2^x\times 2^{-1}+2^x\times 2^1=320$$
$$2^x(2^{-1}+2)=320$$
$$2^x(1/2+2)=320$$
$$2^x(5/2)=320$$
$$2^x=320(2/5)$$
$$2^x=128$$
$$2^x=2^7$$
$$x=7$$

Ex.15 Find the value of *x* if $5^{x+3}-25^{3x-4}=0$.

Solution: $5^{x+3}-25^{3x-4}=0$

$$5^{x+3}=25^{3x-4}$$
$$5^x5^3=(5^2)^{3x-4}$$
$$5^x5^3=5^{6x}5^{-8}$$
$$\frac{5^3}{5^{-8}}=\frac{5^{6x}}{5^x}$$
$$5^{5x}=5^{11}$$
$$5x=11$$
$$x=11/5$$

Ex.16 If $x=\sqrt[3]{3}+\frac{1}{\sqrt[3]{3}}$ then find the value of $3x^3-9x-10$.

Solution: $x=\sqrt[3]{3}+\frac{1}{\sqrt[3]{3}}=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$

$$x=3^{\frac{1}{3}}+3^{-\frac{1}{3}}$$
$$x^3=\left(3^{\frac{1}{3}}+3^{-\frac{1}{3}}\right)^3 \qquad (x+y)^3=x^3+y^3+3xy(x+y)$$
$$x^3=(3^{\frac{1}{3}})^3+(3^{-\frac{1}{3}})^3+3(3^{\frac{1}{3}})(3^{-\frac{1}{3}})(3^{\frac{1}{3}}+3^{-\frac{1}{3}})$$
$$x^3=3+3^{-1}+3(3^{\frac{1}{3}-\frac{1}{3}})x$$
$$x^3=3+3^{-1}+3x$$
$$3x^3=3(3+3^{-1}+3x) \qquad \text{[by multiplying with 3 on both sides]}$$
$$3x^3=9+1+9x$$
$$3x^3=10+9x$$
$$3x^3-9x-10=0$$

Ex.17 If $2^x=3^y=12^z$ then show that $\frac{2}{x}+\frac{1}{y}=\frac{1}{z}$.

Solution: Let us assume that,

$$2^x = 3^y = 12^z = k$$

$$\therefore \quad 2^x = k \Rightarrow 2 = k^{\frac{1}{x}} \Rightarrow 2^2 = k^{\frac{2}{x}} \Rightarrow 4 = k^{\frac{2}{x}}$$

$$\therefore \quad 3^y = k \Rightarrow 3 = k^{\frac{1}{y}}$$

$$\therefore \quad 12^z = k \Rightarrow 12 = k^{\frac{1}{z}}$$

$$4 \times 3 = k^{\frac{2}{x}} \times k^{\frac{1}{y}}$$

$$12 = k^{\frac{2}{x}+\frac{1}{y}} \quad \text{.....(i)}$$

But $\quad 12 = k^{\frac{1}{z}} \quad$ .....(ii)

By comparing (i) and (ii), we have

$$k^{\frac{2}{x}+\frac{1}{y}} = k^{\frac{1}{z}}$$

$$\frac{2}{x} + \frac{1}{y} = \frac{1}{z}$$

:: EXERCISE 1 ::

1. What is mean by exponent. State the rule of exponents.

2. Simplify the following values

(i) $\sqrt[4]{256}$ (ii) $\sqrt[6]{64^{-5}}$ (iii) $\sqrt[4]{81^3}$

3. Simplify the following values

(i) $\sqrt[3]{8^4}$ (ii) $\sqrt[4]{16^3}$

4. Simplify $\dfrac{3^5(3^3)^3 9^4}{3(3^4)^4}$

5. Simplify $\dfrac{4^3(4^2)^2 64^2}{16 \times (2^4)^4}$

6. Simplify $5^{x+2} - (25)(5^2)^{x-1} = 0$

7. Simplify $\dfrac{\sqrt{98} - \sqrt{50}}{\sqrt{6}}$

8. Show that $\left(\dfrac{5^{-1} \cdot 7^2}{5^2 \cdot 7^{-4}}\right)^{\frac{7}{2}} \times \left(\dfrac{5^{-2} \cdot 7^3}{5^3 \cdot 7^{-5}}\right)^{-\frac{5}{2}} = 175$

9. Show that, $\dfrac{2^n + 2^{n-1}}{2^{n+1} - 2^n} = \dfrac{3}{2}$

10. Simplify (i) $\dfrac{x^{3m-8n} \cdot x^{2n+m}}{x^{5m-6n}}$ (ii) $\dfrac{7^{m+2} - 35 \cdot 7^{m-1}}{11 \cdot 7^m}$ (iii) $\dfrac{9^m \times 3^2 \times 3^m - 27^m}{3^2 \times 3^{3m}}$

11. Simplify (i) $\left(\dfrac{x^c}{x^a}\right)^b \left(\dfrac{x^b}{x^c}\right)^a \left(\dfrac{x^a}{x^b}\right)^c$ (ii) $\left(\dfrac{x^c}{x^a}\right)^{1/ac} \left(\dfrac{x^b}{x^c}\right)^{1/bc} \left(\dfrac{x^a}{x^b}\right)^{1/ab}$

12. If $3^x = 5^y = (75)^z$ then show that $xy = z(2x + y)$.

13. If $2^x = 3^y = 6^{-z}$ then show that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$.

14 If $x=\sqrt[3]{3^2}+\frac{1}{\sqrt[3]{3^2}}$ then find the value of $9x^3-27x-82$.

:: ANS. 1 ::

2. (i) 4 (ii) 32 (iii) 27 **3.** (i) 16 (ii) 8 **4.** 243 **5.** 64 **6.** $x = 2$ **7.** $\frac{2}{\sqrt{3}}$ **10.** (i) x^{-m} (ii) 4 (iii) 8/9 **11.** (i) 1 (ii) 1 **14.** 0

1.4 Surds

Definition: The irrational root of a rational number of types $\sqrt[n]{x}\,(=\sqrt[3]{7})$, where it is not possible to extract exactly the n^{th} root of *x*, is called surd.

A real number $\sqrt[n]{x}$ is called the surd, if and only if it satisfies the following conditions:

1. It is an irrational number, i.e. $\sqrt[n]{x}$ is an irrational number.
2. It is a root of a rational number, i.e. *x* is a rational number

In the surd $\sqrt[n]{x}$, the **index *n*** is called the **order of the surd** and ***x*, the radicand**.

For example: (i) $\sqrt{7}$ is called a surd, since $\sqrt{7}$ is an irrational root of the rational number.

(ii) $\sqrt[4]{12}$, $\sqrt[3]{11}$, $\sqrt[6]{32}$ are surds.

(iii) $\sqrt[3]{27}$, is not surds because its root is rational, i.e. 3.

(iv) $\sqrt[3]{\frac{64}{27}}$, is not surds because its root is rational, i.e. 4/3.

(v) $\sqrt[3]{5+\sqrt{3}}$, is not surds because radicand of it, is not rational number.

[1] Order of a surd

The order of surd is number which indicates the value of **index *n***.

For example: (i) $\sqrt{7}$ is square root of 7, i.e. second order surd.

(ii) $\sqrt[3]{11}$, is cube root of 11, i.e. third order surd.

(iii) $\sqrt[6]{32}$ is sixth root of 32, i.e. sixth order surd.

(iv) $\sqrt[n]{x}$ is n^{th} root of *x*, i.e. *n*th order surd.

[2] Entire and Mixed surd

A surd without coefficient is called entire surd.

For example: $\sqrt{7}$ is entire surd.

A surd with coefficient is called mixed surd.

For example: $4\sqrt{7}$ is mixed surd.

Entire surd can be written as mixed surd also or vice versa.

For example: $\sqrt{112}$ can be written as $\sqrt{112}=\sqrt{16\times 7}=4\sqrt{7}$.

1.5 Operations With Surds

Surds can always be expressed with fractional indices. Therefore, the rules of indices given earlier will apply to surds also.

[1] Surds of the same order can be multiplied as under

$$\sqrt[n]{x} \times \sqrt[n]{y} = \sqrt[n]{xy} \quad \text{or} \quad x^{\frac{1}{n}} \times y^{\frac{1}{n}} = (x \times y)^{\frac{1}{n}}$$

Ex. 18 Simplify (i) $\sqrt[3]{2} \times \sqrt[3]{5} = \sqrt[3]{2 \times 5} = \sqrt[3]{10}$

(ii) $4\sqrt[2]{6} \times 3\sqrt[2]{7} = (4 \times 3)\sqrt[2]{6 \times 3}$

$= 12\sqrt[2]{18}$

$= 12\sqrt[2]{9 \times 2}$

$= (12 \times 3)\sqrt[2]{2}$

$= 36\sqrt[2]{2}$

[2] Surds of the same order can be divided as under

$$\sqrt[n]{x} \div \sqrt[n]{y} = \sqrt[n]{\frac{x}{y}} \quad \text{or} \quad x^{\frac{1}{n}} \div y^{\frac{1}{n}} = \left(\frac{x}{y}\right)^{\frac{1}{n}}$$

Ex. 19 Simplify (i) $\sqrt[3]{2} \div \sqrt[3]{5} = \sqrt[3]{\frac{2}{5}}$

(ii) $10\sqrt[3]{7} \div 5\sqrt[3]{5} = \frac{10}{5}\sqrt[3]{\frac{7}{5}} = 2\sqrt[3]{\frac{7}{5}}$

(iii) $10\sqrt[3]{9} \div 5\sqrt[6]{3} = 10\sqrt[3\times2]{3^2} \div 5\sqrt[6]{3}$

$= 10\sqrt[3\times2]{9^2} \div 5\sqrt[6]{3}$

$= 10\sqrt[6]{81} \div 5\sqrt[6]{3}$

$= \frac{10}{5}\sqrt[6]{\frac{81}{3}} = 2\sqrt[6]{27}$

Ex. 20 Simplify $\frac{\sqrt{72} - \sqrt{32}}{\sqrt{18}}$

Solution : $\frac{\sqrt{72} - \sqrt{32}}{\sqrt{18}} = \frac{\sqrt{36 \times 2} - \sqrt{16 \times 2}}{\sqrt{9 \times 2}}$

$= \frac{6\sqrt{2} - 4\sqrt{2}}{3\sqrt{2}}$

$= \frac{\sqrt{2}(6-4)}{3\sqrt{2}} = \frac{2}{3}$

[3] Surds of the different order can be reduced to the lowest possible common order by multiplying both the index and radicand by the same as under:

$$\sqrt[m]{x^n} = \sqrt[mp]{x^{np}} \quad \text{or} \quad x^{\frac{1}{n}} = (x^p)^{\frac{1}{pn}}$$

For example: $\sqrt[3]{2}$ can be written as $\sqrt[3\times2]{2^2} = \sqrt[6]{4}$

[4] Surds of the different order can be multiplied as under

$$\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[m \times n]{a^n \times b^m}$$

Ex. 21 Simplify $\sqrt[3]{2} \times \sqrt[4]{3}$

Solution: $\sqrt[3]{2} \times \sqrt[4]{3} = \sqrt[3\times4]{2^4 \times 3^3}$ $\quad [\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[m \times n]{a^n \times b^m}]$

$= \sqrt[12]{16 \times 9}$

$= \sqrt[12]{144}$

[5] Addition and Subtraction of Surds are as under

(i) $a\sqrt[n]{x} + b\sqrt[n]{x} = (a+b)\sqrt[n]{x}$

(ii) $a\sqrt[n]{x} - b\sqrt[n]{x} = (a-b)\sqrt[n]{x}$

For example: (i) $3\sqrt[3]{2} + 5\sqrt[3]{2} = (3+5)\sqrt[3]{2} = 8\sqrt[3]{2}$

(ii) $6\sqrt[3]{3} - 3\sqrt[3]{3} = (6-3)\sqrt[3]{3} = 3\sqrt[3]{3}$

Ex.22 Simplify $6\sqrt{80} - 2\sqrt{20} + 3\sqrt{45}$

Solution : $6\sqrt{80} - 2\sqrt{20} + 3\sqrt{45} = 6\sqrt{16\times5} - 2\sqrt{4\times5} + 3\sqrt{4\times5}$

$= 6\times4\sqrt{5} - 2\times2\sqrt{5} + 3\times2\sqrt{5}$

$= 24\sqrt{5} - 4\sqrt{5} + 6\sqrt{5}$

$= 26\sqrt{5}$

Ex.23 Solve the equation $\sqrt{x-4} = \sqrt{x+11} - 3$.

Solution: $\sqrt{x-4} = \sqrt{x+11} - 3$

by squaring on both sides, we get

$(\sqrt{x-4})^2 = (\sqrt{x+11} - 3)^2$

$x - 4 = 9 - 6\sqrt{x+11} + x + 11$

$24 = 6\sqrt{x+11}$

$4 = \sqrt{x+11}$

$(4)^2 = (\sqrt{x+11})^2$

$16 = x + 11$

$x = 5$

Ex.24 If $x = \sqrt{7} + \sqrt{8}$, then find the value of $\frac{x+1}{x-1} + \frac{x-1}{x+1}$.

Solution:

$$\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{(x+1)(x+1) + (x-1)(x-1)}{(x-1)(x+1)}$$

$$= \frac{(x+1)^2 + (x-1)^2}{x^2 - 1}$$

$$= \frac{x^2 + 2x + 1 + x^2 - 2x + 1}{x^2 - 1}$$

$$= \frac{2x^2 + 2}{x^2 - 1} = \frac{2(x^2+1)}{x^2 - 1}$$

Now $\quad x = \sqrt{7} + \sqrt{8}$

$x^2 = (\sqrt{7} + \sqrt{8})^2$

$$x^2 = 7 + 2\sqrt{56} + 5$$
$$x^2 = 12 + 2\sqrt{14 \times 4}$$
$$x^2 = 12 + 4\sqrt{14}$$
$$x^2 - 1 = 12 + 4\sqrt{14} - 1 \qquad \text{.....(i)}$$
$$x^2 - 1 = 11 + 4\sqrt{14}$$
$$x^2 + 1 = 12 + 4\sqrt{14} + 1$$
$$x^2 + 1 = 13 + 4\sqrt{14} \qquad \text{.....(ii)}$$
$$\frac{2(x^2+1)}{x^2-1} = \frac{2(13+4\sqrt{14})}{11+2\sqrt{14}} \qquad \text{[From (i) and (ii)]}$$

1.6 Rationalising Factor

When the product of two surds is rational then each one of them is called rationalising factor of the other.

[1] Rationalising for Monomial Surds: Rationalising factor of Monomial Surds is obtained as under:

(i) $a\sqrt{x}$ can be rationalize by multiplying with $\sqrt{x}$, we have

$$a\sqrt{x} \times \sqrt{x} = ax$$

(ii) $4\sqrt{2}$ can be rationalize by multiplying with $\sqrt{2}$, we have

$$4\sqrt{2} \times \sqrt{2} = 4 \times 2 = 8$$

[2] Rationalising for Binomial Surds: Rationalising factor of Binomial Surds is obtained as under:

(i) $\sqrt{x} + \sqrt{y}$ can be rationalize by multiplying with $\sqrt{x} - \sqrt{y}$, we have

$$(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x - y$$

(ii) $\sqrt{2} + \sqrt{3}$ can be rationalize by multiplying with $\sqrt{2} - \sqrt{3}$, we have

$$(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3}) = 2 - 3 = -1$$

Ex.25 Rationalise $\frac{5-\sqrt{24}}{5+\sqrt{24}}$

Solution: $\frac{5-\sqrt{24}}{5+\sqrt{24}}$

By multiplying and dividing with $5 - \sqrt{24}$, we get

$$\frac{5-\sqrt{24}}{5+\sqrt{24}} = \frac{5-\sqrt{24}}{5+\sqrt{24}} \times \frac{5-\sqrt{24}}{5-\sqrt{24}}$$
$$= \frac{(5-\sqrt{24})^2}{25-24}$$
$$= \frac{25 - 10\sqrt{24} + 24}{1}$$
$$= 49 - 10\sqrt{24}$$

Ex.26 Simplify $\frac{3\sqrt{7}}{\sqrt{8}+\sqrt{7}}-\frac{4\sqrt{7}}{\sqrt{10}+\sqrt{6}}$.

Solution: By rationalization, we get

$$\frac{3\sqrt{7}}{\sqrt{8}+\sqrt{7}}-\frac{4\sqrt{7}}{\sqrt{10}+\sqrt{6}}=\frac{3\sqrt{7}}{\sqrt{8}+\sqrt{7}}\times\frac{\sqrt{8}-\sqrt{7}}{\sqrt{8}-\sqrt{7}}-\frac{4\sqrt{7}}{\sqrt{10}+\sqrt{6}}\times\frac{\sqrt{10}-\sqrt{6}}{\sqrt{10}-\sqrt{6}}$$

$$=\frac{3\sqrt{7}(\sqrt{8}-\sqrt{7})}{8-7}-\frac{4\sqrt{7}(\sqrt{10}-\sqrt{6})}{10-6}$$

$$=\frac{3\sqrt{7}(\sqrt{8}-\sqrt{7})}{1}-\frac{4\sqrt{7}(\sqrt{10}-\sqrt{6})}{4}$$

$$=3\sqrt{7}\sqrt{8}-3\sqrt{7}\sqrt{7}-\sqrt{7}\sqrt{10}-\sqrt{7}\sqrt{6}$$

$$=3\sqrt{56}-3\times7-\sqrt{70}-\sqrt{42}$$

$$=3\sqrt{56}-21-\sqrt{70}-\sqrt{42}$$

Ex.27 If $x=4+\sqrt{15}$, find the value of $x^2-\frac{1}{x^2}$

Solution: If $x=4+\sqrt{15}$ then $\frac{1}{x}=\frac{1}{4+\sqrt{15}}$

$$\frac{1}{x}=\frac{1}{4+\sqrt{15}}\times\frac{4-\sqrt{15}}{4-\sqrt{15}} \qquad \text{[By rationalization]}$$

$$\frac{1}{x}=\frac{1}{4+\sqrt{15}}\times\frac{4-\sqrt{15}}{4-\sqrt{15}}$$

$$\frac{1}{x}=\frac{4-\sqrt{15}}{(4+\sqrt{15})(4-\sqrt{15})}$$

$$\frac{1}{x}=\frac{4-\sqrt{15}}{16-15} \quad \frac{1}{x}=4-\sqrt{15}$$

$$x+\frac{1}{x}=(4+\sqrt{15})+(4-\sqrt{15})=8 \qquad \text{.....(i)}$$

$$x-\frac{1}{x}=(4+\sqrt{15})-(4-\sqrt{15})=2\sqrt{15} \qquad \text{.....(ii)}$$

Now, $x^2-\frac{1}{x^2}=(x+\frac{1}{x})(x-\frac{1}{x})$ [From (i) and (ii)]

$$=8(2\sqrt{15}) \quad =16\sqrt{15}$$

Ex.28 If $x=\frac{\sqrt{17}-4}{\sqrt{17}+4}$, find the value of $x^2+\frac{1}{x^2}$

Solution: $x=\frac{\sqrt{17}-4}{\sqrt{17}+4}$

$$x=\frac{\sqrt{17}-4}{\sqrt{17}+4}\times\frac{\sqrt{17}-4}{\sqrt{17}-4} \qquad \text{[By rationalization]}$$

$$= \frac{(\sqrt{17}-4)^2}{17-16}$$

$$= \frac{17-8\sqrt{17}+16}{17-16}$$

$$x = \frac{33-8\sqrt{17}}{1} = 33-8\sqrt{17} \qquad .....(i)$$

$$\frac{1}{x} = \frac{\sqrt{17}+4}{\sqrt{17}-4}$$

$$\frac{1}{x} = \frac{\sqrt{17}+4}{\sqrt{17}-4} \times \frac{\sqrt{17}+4}{\sqrt{17}+4}$$ [By rationalization]

$$= \frac{(\sqrt{17}+4)^2}{17-16}$$

$$= \frac{17+8\sqrt{17}+16}{17-16}$$

$$= \frac{33+8\sqrt{17}}{1} = 33+8\sqrt{17} \qquad .....(ii)$$

$$x+\frac{1}{x} = 33-8\sqrt{17}+33+8\sqrt{17}$$ [From (i) and (ii)]

$$x+\frac{1}{x} = 66$$

$$\left(x+\frac{1}{x}\right)^2 = (66)^2$$ [By squaring both sides]

$$x^2+2x\frac{1}{x}+\frac{1}{x^2} = (66)^2$$

$$x^2+2+\frac{1}{x^2} = 4356$$

$$x^2+\frac{1}{x^2} = 4354$$

Ex.29 If $x = 5+2\sqrt{6}$, and $y = \frac{1}{5+2\sqrt{6}}$ find the value of $3x^2+6xy+3y^2$.

Solution: $x = 5+2\sqrt{6}$

$$x^2 = (5+2\sqrt{6})^2$$ [By squaring both sides]

$$x^2 = 25+20\sqrt{6}+24$$

$$x^2 = 49+20\sqrt{6}$$

$$y = \frac{1}{5+2\sqrt{6}}$$

$$y = \frac{1}{5+2\sqrt{6}} \times \frac{5-2\sqrt{6}}{5-2\sqrt{6}}$$ [By rationalization]

$$y = \frac{5-2\sqrt{6}}{25-24}$$

$$y = 5-2\sqrt{6}$$

$$y^2 = (5-2\sqrt{6})^2 \qquad \text{[By squaring both sides]}$$

$$y^2 = 25-20\sqrt{6}+24$$

$$y^2 = 49-20\sqrt{6}$$

$$x\,y = 5+2\sqrt{6} \times \frac{1}{5+2\sqrt{6}} = 1$$

$$3x^2+6xy+3y^2 = 3(49+20\sqrt{6})+6(1)+3(49-20\sqrt{6})$$

$$3x^2+6xy+3y^2 = 147+60\sqrt{6}+6+147-60\sqrt{6}$$

$$3x^2+6xy+3y^2 = 300$$

[3] Roots of Mixed Surds

If $x \pm \sqrt{y} = a \pm \sqrt{b}$ then $a = x$ and $b = y$

where *a*, *x* are rational number and $\sqrt{y}$ and $\sqrt{b}$ are surds.

If $\sqrt{x+\sqrt{y}} = \sqrt{a}+\sqrt{b}$ then its roots are written as

$$\sqrt{x-\sqrt{y}} = \pm(\sqrt{a}-\sqrt{b})$$

Where *a*, *b*, *x* are rational number and $\sqrt{y}$ is surds.

Ex.30 Find the square root of $3+\sqrt{6}$ or Find the value of $\sqrt{3+\sqrt{6}}$.

Solution: Let $\sqrt{3+\sqrt{5}} = \sqrt{a}+\sqrt{b}$

$$3+\sqrt{5} = a+b+2\sqrt{ab}$$

$$a+b=3 \qquad \text{…..(i)}$$

$$2\sqrt{ab} = \sqrt{5}$$

$$4ab = 5$$

$$ab = 5/4 \qquad \text{…..(ii)}$$

$$(a-b)^2 = (a+b)^2 - 4ab$$

$$(a-b)^2 = (3)^2 - 4(5/4) \qquad \text{[From (i) and (ii)]}$$

$$(a-b)^2 = 9-5$$

$$a-b=4$$

By solving (i) and (ii), we get

a = 5/2 and *b* = 1/2

Hence, $\sqrt{3+\sqrt{5}} = \sqrt{a}+\sqrt{b}$

$$\sqrt{3+\sqrt{5}} = \sqrt{5/2}+\sqrt{1/2}$$

:: EXERCISE 2 ::

1. What is mean by Surd. State the rule of Surd.

2. Simplify the following values

(i) $\sqrt{112}-\sqrt{63}+224/\sqrt{28}$ (ii) $\sqrt{63}+\sqrt{28}-\sqrt{175}$

3. Simplify the following term

$$\frac{3+\sqrt{6}}{5\sqrt{3}+\sqrt{50}-\sqrt{32}-2\sqrt{12}}$$

4. Simplify $\frac{7-3\sqrt{5}}{3-\sqrt{5}}+\frac{7+3\sqrt{5}}{3+\sqrt{5}}$

5. Simplify $\frac{7\sqrt{3}}{\sqrt{10}+\sqrt{3}}-\frac{2\sqrt{5}}{\sqrt{6}+\sqrt{5}}-\frac{3\sqrt{2}}{\sqrt{15}+3\sqrt{2}}$

6. Find the value of *a* and *b* if both are rational numbers and

$$\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a-b\sqrt{3}$$

7. Find the value of *a* and *b* if both are rational numbers and

$$\frac{3-\sqrt{5}}{3+2\sqrt{5}}=a\sqrt{5}-b$$

8. If $x=3+\sqrt{2}$ find the value of $x^4-4x^3-2x^2-4x+31$.

9. If $x=\frac{1}{5+2\sqrt{6}}$ find the value of $x^2-10x+1$.

10 If $x=\sqrt{3}+\sqrt{2}$, then find the value of $\frac{x+1}{x-1}+\frac{x-1}{x+1}$.

11. If $x=7+4\sqrt{3}$, and $y=7-4\sqrt{3}$ find the value of $x^{-2}+y^{-2}$.

12. If $x=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, and $y=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ find the value of x^2+y^2.

13. If $x=9+4\sqrt{5}$, and $y=\frac{1}{9+4\sqrt{5}}$ find the value of $5x^2+10xy+5y^2$.

:: ANS. 2 ::

2. (i) $17\sqrt{7}$ (ii) 0 **3.** $\sqrt{3}$ **4.** 3 **5.** 1 **6.** *a* = 11, *b* = 6

7. *a* = 9/11, *b* = 19/11 **8.** 10 **9.** 0 **10.** $\sqrt{6}$ **11.** 194

12. 98 **13.** 1180

 Select the appropriate answer from the given alternative answer. (M.C.Q.)

1. If $\left(\frac{a}{b}\right)^{x-1}=\left(\frac{b}{a}\right)^{x-3}$ then value of *x* is

(a) 1 (b) 2 (c) 3 (d) 4

2. If $(16)^{\frac{7}{2}}\times(16)^{x}=(16)^{8}$ then value of *x* is

(a) 2.5 (b) 3 (c) 4.5 (d) 5.5

3. If $5^{a} = 3125$ then value of $5^{a-3} =$

(a) 25 (b) 125 (c) 250 (d) 625

4. If $3^{x-y} = 27$ and $3^{x+y} = 243$ then value of *x* is

(a) 1 (b) 2 (c) 3 (d) 4

5. Value of $10^{150} \div 10^{148} =$

(a) 2 (b) 10 (c) 100 (d) 1000

6. If $(25)^{\frac{15}{2}} \times (5)^{\frac{5}{2}} \div (125)^{\frac{3}{2}} = 5^{x}$ then value of *x* is

(a) 12 (b) 13 (c) 14 (d) 15

7. Value of $(-1)^{-1} =$

(a) –1 (b) 1 (c) –2 (d) 0

8. Value of $(6)^{x} \div (6)^{-3} = (6)^{5}$

(a) –1 (b) 1 (c) –2 (d) 2

9. Value of $(100 - (99)^{0}) \times 100 =$

(a) 100 (b) 990 (c) 9900 (d) 10000

10. Value of $\left(\frac{4}{5}\right)^{-8} \times \left(\frac{5}{4}\right)^{-8} =$

(a) –1 (b) 1 (c) –2 (d)

:: ANS. ::

1. (b) **2.** (c) **3.** (a) **4.** (d) **5**. (b)

6. (b) **7.** (a) **8.** (d) **9.** (c) **10.** (b)

UNIT-2 SET THEORY

2.1 Introduction

The theory of set was developed by German Mathematician Georg Cantor
(1845 – 1918). He is regarded as the father of set theory. We shall discuss some of the fundamental definitions and operations involving sets.

"A set is a collection of well-defined and well distinguished objects."

For e.g. (i) Number of students in F.Y. B.B.A. class

(ii) Number of vowels in English alphabet

(iii)Number of subjects in F. Y. B. B. A.

From the above example we can say the "Set" is a homogeneous group of objects or articles having the same characteristics. The objects or articles that make up a set are called the members or elements of the set. A set is denoted by capital letters A, B, C,... and its elements are denoted by lower case letters a, b, c,....

Symbol used in the set theory are as under :

(i) Epsilon : $\in$ "a members of the set"

(ii) Phi : ϕ

(iii) Union : $\cup$

(iv) Intersection : $\cap$

(v) Equivalent : $\equiv$

(vi) Subset : $\subset$

(vii) Every element : $\forall$

2.2 Methods Of Representation Of Set

The set is a collection of objects having similar characteristics, can be represented by two methods, namely (a) Roster Method and (b) Rule Method.

(a) Roster Method / Listing Method/Tabular Form :

In Roster method set can be represented by showing all the members within bracket. We make a list of the elements of set and put it within braces.

E.g.. (i) A set of vowels : A = {a, e, i, o, u}

(ii) A set of even natural numbers : E = {2, 4, 6, 8,.....}

(iii) A set of integer : I = {1, 2, 3, 4,.....}

(b) Rule Method / Set Builder Form / Property Method :

In Rule method, we list the property or properties satisfied by the element of set.

For e.g. consider A be a set of all positive integer less than 4.

Set A contains 1, 2, 3 as the member or elements:

$\therefore$ A = {x / 0 < x < 4, x $\in$ N}

For e.g. Let B be the set of all odd positive integers less than 10.

It means that 1, 3, 5, 7, 9 are the members or elements of B.

$\therefore$ B = {x / x is a positive odd integer and x < 10}.

Note : '/' Horizontal line for "such that".

2.3 Types of Set

In practice different types of set are used. They are defined as under:

[1] Universal Set :

The universal set in any discussion is the totality of members under consideration as elements of set. Universal set is denoted by U.

For example : U = {1, 2, 3, 4,.......}

E = {2, 4, 6, 8,......}

O = {1, 3, 5,.....}

Here, E and O are the subsets of universal set U.

[2] Finite Set/Infinite Set :

A set is called finite if it has a finite number of elements; otherwise, it is called infinite.

For example : (i) Number of states in India

(ii) Number of districts in Gujarat.

(iii) Number of students in Central University.

From the above example, it is clear that number of members in a set is countable and finite.

Let N = Set of all natural numbers.

$\therefore$ N = {1, 2, 3,........}

Here we cannot determine the last element of set.

$\therefore$ N is an infinite set.

Let $A = \left\{ \frac{1}{x} / x \in N \right\}$

It may also be described as

$$A = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ... \right\}$$

There is no finite number of elements.

$\therefore$ A is an infinite set.

Some times the number of elements in the set is large but finite, then that set is finite set.

For example : A = Set of all positive integers divisible by 5 and less than 950.

$\therefore$ The last element of set is 950.

Set A has a finite number of elements, say, 190. Hence A is a finite set.

[3] Singleton Set :

A set having only one element is known as singleton set.

e.g. A = {1}

[4] Empty set / Null set :

The set without an element is known as empty set or null set.

It is denoted by ϕ or { }.

For example : (i) Set of living persons whose age more than 150 years.

(ii) The set of all prime number between 19 and 23 is an empty set.

[5] Equal Set :

Two sets A and B are said to be equal if each element of A is an element of B and each element of B is also an element of A and same is written as, A = B.

If ($\forall x / x \in A \Rightarrow x \in B$) and ($\forall x / x \in B \Rightarrow x \in A$), then A = B.

Also, $A \subset B$, $B \subset A \Rightarrow A = B$.

For example : (i) A = {1, 2, 3}, B = {1, 2, 3}

(ii) A = {2, –2}, $B = \{x / x^2 = 4, x \in Z\}$

$x^2 = 4$

x = 2 or –2

B = {2, –2}

$\therefore$ A = B

[6] Equivalent Set :

Two sets A and B are said to be equivalents if, it is possible to match the elements of A with the elements of B such that each element of A is matched with exactly one element of B.

For example : $A = \{1, 2, 3\}$ or $A = \{x / 0 < x < 4, x \in N\}$

$B = \{1, 4, 9\}$ or $B = \{x^2 / 0 < x < 4, x \in N\}$

Set A		Set B
1	$\leftrightarrow$	1
2	$\leftrightarrow$	4
3	$\leftrightarrow$	9

By matching members as above, we get one to one correspondence.

Here, set A is equivalent to set B.

Symbolically we write as: $A \leftrightarrow B$ or $A \equiv B$.

[7] Subsets :

The set A is the subset of set B if, each element of set A is an element of set B. Set B is called super set.

Thus, $(\forall x / x \in A \Rightarrow x \in B)$ then $A \subset B$.

In notation, $A \subset B$ if and only if, $x \in A \Rightarrow x \in B$.

For example:

Let $A = \{1, 3\}$ and $B = \{1, 2, 3\}$

Here every element of set A is also member of set B but every element of set B is not member of set A. That is set A is included in set B.

$\therefore$ Set A is subset of set B. i.e. $A \subseteq B$.

Note : (i) Every set is subset of that set itself, i.e. for set A , $A \subseteq A$.

(ii) Null set is subset of every set, i.e. for set A, $\phi \subset A$.

(iii) There are 2^n, $n \in N$, subset of every set.

(a) $A = \{1, 2\}$ Number of subsets $= 2^2 = 4$

$\therefore$ Subset of set A are $\phi, \{1\}, \{2\}, \{1, 2\}$.

(b) $A = \{1, 2, 3\}$ Number of subsets $= 2^3 = 8$

$\therefore$ Subset of set $A = \phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$.

(iv) If $A \subset B$ and $B \subset C$ then $A \subset C$

[8] Proper Subset :

If all elements of set A are the elements of set B and at least one element of super set B is not an element of set A, then set A is called proper subset of super set B.

For example: A = {1, 3, 5}, B = {1, 3, 5, 7}.   Here $A \subset B$.

[9] Family of Set :

If all the elements of a set are sets themselves, then it is called a family of sets.

For example, if A = {1, 2} then the set B = {0, {1},{1}, {1, 2}} is a family of sets whose elements are subsets of the set A.

[10] Power Set :

The family of all the subsets of a given set is called the power set.

The power set of a set is the set of all the subsets of the given set.

E.g. If A = {a, b, c} then its power set denoted by P(A) can be given by,

P(A) = {ϕ , {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

Note : The set having n elements will have 2^n subset in its power set.

2.4 Venn Diagrams

The Venn diagrams are pictorial diagrams representing the different sets and its elements. The Venn diagrams are named after English logician John Venn (1834 - 1883).

In Venn Diagram, the Universal set, is denoted by U a region enclosed by a rectangle and its sets, say, A, B, C,... are shown through circles within these rectangles.

Venn Diagram is very useful to see the different set relations, such as intersection of sets, union of sets, complementation of set, difference sets etc.

2.5 Operations On Sets

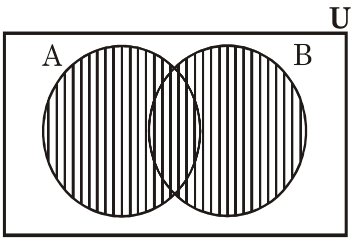
[1] Union of set :

"The union of two sets A and B is the set consisting of all the elements which belong to either A or B or both".

The union of set A and B is denoted by $A \cup B$, read as A union B.

Symbolically, $A \cup B = \{ x / x \in A \text{ or } x \in B \text{ or } x \in \text{both A and B} \}$

Union of two sets can be shown (shaded region) by Venn Diagram as under



Properties of Union of sets.

(i) $A \subseteq (A \cup B)$ and $B \subseteq (A \cup B)$

(ii) $A \cup \phi = A$, for every set A. [Existence of identity ϕ]

(iii) $A \cup A = A$ for every set A.

(iv) $A \cup B = B \cup A$ [Commutative property]

(v) $(A \cup B) \cup C = A \cup (B \cup C)$ [Associative property]

(vi) If $B \subseteq A$ then $A \cup B = A$ and if $A \subseteq B$, then $A \cup B = B$.

(vii) $A \cup B = \phi \Rightarrow A = \phi$ and $B = \phi$

(viii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(Distributive Law of Union over Intersection)

(ix) If $A \subset B$ and $C \subset D$ then $(A \cup C) \subset (B \cup D)$

Theorem 1. Prove that $(A \cup B) \cup C = A \cup (B \cup C)$. [Associative Property]

Proof : Let x be any element of $(A \cup B) \cup C$.

Then $x \in (A \cup B) \cup C \Rightarrow (x \in (A \cup B))$ or $(x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in (A \cup (B \cup C))$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$ ..... (1)

Let y be any element of $A \cup (B \cup C)$.

$\therefore y \in A \cup (B \cup C) \Rightarrow y \in A$ or $y \in (B \cup C)$

$\Rightarrow y \in A$ or $(y \in B$ or $y \in C)$

$\Rightarrow (y \in A$ or $y \in B)$ or $y \in C$

$\Rightarrow y \in (A \cup B)$ or $y \in C$

$\Rightarrow y \in (A \cup B) \cup C$

$\therefore A \cup (B \cup C) \subseteq (A \cup B) \cup C$ ...... (2)

From (1) and (2), by definition of equality of sets, we have

$(A \cup B) \cup C = A \cup (B \cup C)$.

Ex.1 Let A = {1, 2, 3}, B = {3, 4}, C = {4, 5, 6} find (i) $A \cup B$ (ii) $A \cup C$ (iii) $A \cup (B \cup C)$

Solution :

(i) $A \cup B = \{1, 2, 3\} \cup \{3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

(ii) $A \cup C = \{1, 2, 3\} \cup \{4, 5, 6\}$

$A \cup C = \{1, 2, 3, 4, 5, 6\}$

(iii) $A \cup (B \cup C) = \{1, 2, 3\} \cup (\{3, 4\} \cup \{4, 5, 6\})$

$= \{1, 2, 3\} \cup \{3, 4, 5, 6\}$

$A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6\}$

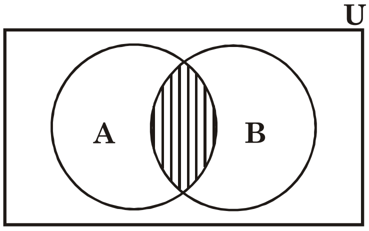
[2] Intersection of Sets :

"The intersection of two sets A and B is the set consisting of all elements which belong to both A and B".

The intersection of A and B is denoted by $A \cap B$.

Symbolically, $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Intersection of two sets can be shown (shaded region) by Venn Diagram as under:



Properties of Intersection of sets.

(i) $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

(ii) $A \cap \phi = \phi$ for every set A.

(iii) $A \cap A = A$ for every set A.

(iv) $A \cap B = B \cap A$ (Commutative property)

(v) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative property)

(vi) If $B \subseteq A$ then $A \cap B = B$ and if $A \subseteq B$ then $A \cap B = A$.

(vii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq (B \cap C)$.

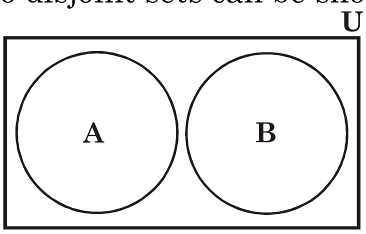
(viii) If $A \subset B$ and $C \subset D$ then $(A \cap C) \subset (B \cap D)$

(ix) Distributive Law of Intersection over Union

$A \cap (B \cup C) = (A \cap B) \cup ((A \cap C)$

Disjoint Sets : Any two sets are said to be disjoint sets if their intersection is the null set. A and B are said to be disjoint if $A \cap B = \phi$.

Two disjoint sets can be shown by Venn Diagram as under:



Theorem 2 Prove that $(A \cap B) \cap C = A \cap (B \cap C)$. [Associative Property]

Proof : Let $x \in (A \cap B) \cap C \Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ and } (x \in C)$

$\Rightarrow (x \in A) \text{ and } (x \in B \text{ and } x \in C)$

$\Rightarrow x \in A \text{ and } x \in (B \cap C)$

$\Rightarrow x \in A \cap (B \cap C)$

$\therefore (A \cap B) \cap C \subseteq A \cap (B \cap C)$ ..... (1)

Let y be any element of $A \cap (B \cap C)$

Then, $y \in A \cap (B \cap C) \Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow (y \in A)$ and $(y \in B$ and $y \in C)$

$\Rightarrow (y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow (y \in A \cap B)$ and $y \in C$

$\Rightarrow y \in (A \cap B) \cap C$

$\therefore A \cap (B \cap C) \subseteq (A \cap B) \cap C.$ ..... (2)

From (1) and (2), by the definition of equality of sets, we have

$(A \cap B) \cap C = A \cap (B \cap C)$

Ex.2 If $A = \{0, 1, 3, 5\}$, $B = \{1, 2, 4, 7\}$, $C = \{1, 2, 3, 5, 8\}$ find the following:
(i) $A \cup B$ (ii) $B \cap C$ (iii) Show that $(A \cap B) \cap C = A \cap (B \cap C)$

Solution : (i) $A \cup B = \{0, 1, 2, 3, 4, 5, 7\}$

(ii) $B \cap C = \{1, 2\}$

(iii) $(A \cap B) \cap C = A \cap (B \cap C)$

L.H.S. = $A \cap B = \{1\}$

$(A \cap B) \cap C = \{1\} \cap \{1, 2, 3, 5, 8\}$

$(A \cap B) \cap C = \{1\}$

R.H.S. = $B \cap C = \{1, 2\}$

$A \cap (B \cap C) = \{0, 1, 3, 5\} \cap \{1, 2\} = \{1\}$

L.H.S. = R.H.S.

Ex.3 If $A = \{x / x \leq 6, x \in N\}$, $B = \{x / 2 \leq x \leq 7, x \in N\}$, $C = \{ x / x \leq 4, x \in N\}$, then find (i) $A \cap B$ (ii) $B \cap C$ (iii) $A \cap C$.

Solution :

Given, $A = \{x / x \leq 6, x \in N\}$

$\therefore A = \{1, 2, 3, 4, 5, 6\}$

$B = \{x / 2 \leq x \leq 7, x \in N\}$

$\therefore B = \{2, 3, 4, 5, 6, 7\}$

$C = \{x / x \leq 4, x \in N\}$

$\therefore C = \{1, 2, 3, 4\}$

(i) $A \cap B = \{2, 3, 4, 5, 6\}$ or $A \cap B = \{y / 2 \leq y \leq 6, y \in N\}$

(ii) $B \cap C = \{2, 3, 4\}$ or $B \cap C = \{y / 2 \leq y \leq 4, y \in N\}$

(iii) $A \cap C = \{1, 2, 3, 4\}$ or $A \cap C = \{y / 1 \leq y \leq 4, y \in N\}$

Theorem 3 Distribution Law of Union over Intersection

Let A, B and C be any 3 sets, prove that,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof : (i) Let x be any element of $A \cup (B \cap C)$.

Let $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ ..... (1)

Let y be any element of $(A \cup B) \cap (A \cup C)$.

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow$ ($y \in A$ or $y \in B$) and ($y \in A$ or $y \in C$)

$\Rightarrow y \in A$ or ($y \in B$ and $y \in C$)

$\Rightarrow y \in A$ or ($y \in B \cap C$)

$\Rightarrow y \in A \cup (B \cap C)$

$\therefore (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ ..... (2)

From (1) and (2) by the definition of equality of sets, we have

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Ex.4 If the universal set is $U = \{x : x \in N, 1 \leq x \leq 12\}$ and

$A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subset of U.

Find $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$.

Solution: (i) $A \cup (B \cap C)$

$(B \cap C) = \{6\}$

$A \cup (B \cap C) = \{1, 6, 9, 10\}$

(ii) $(A \cup B) \cap (A \cup C)$

$A \cup B = \{1, 3, 4, 6, 9, 10, 11, 12\}$

$A \cup C = \{1, 2, 5, 6, 9, 10\}$

$(A \cup B) \cap (A \cup C) = \{1, 6, 9, 10\}$

Ex. 5 If $A = \{x / |x^3 - 2| \leq 25, x \in N\}$, $B = \{x / 2 \leq x \leq 4, x \in N\}$ and

$C = \{x / x^4 = 81, x \in N\}$ then verify that,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution :

$A = \{x / |x^3 - 2| \leq 25, x \in N\}$

$|x^3 - 2| \leq 25$

When x = 1 then $|1^3 - 2| \leq 25 \Rightarrow |1 - 2| \leq 25$, which is true.

x = 2 then $|2^3 - 2| \leq 25 \Rightarrow |8 - 2| \leq 25$, which is true.

x = 3 then $|3^3 - 2| \leq 25 \Rightarrow |27 - 2| \leq 25$, which is true.

$\therefore$ $A = \{1, 2, 3\}$

$B = \{x / 2 \leq x \leq 4, x \in N\}$

$\therefore$ $B = \{2, 3, 4\}$

$C = \{x / x^4 = 81, x \in N\}$

$x^4 = 81 \Rightarrow x^4 = 3^4 \Rightarrow x = 3$

$\therefore$ C = {3}

$\therefore$ A = {1, 2, 3}, B = {2, 3, 4}, C = {3}

LHS = $B \cap C$ = {3}

$\therefore$ $A \cup (B \cap C)$ = {1, 2, 3} $\cup$ {3}

= {1, 2, 3} ..... (1)

RHS = $A \cup B$ = {1, 2, 3, 4}

$A \cup C$ = {1, 2, 3}

$\therefore$ $(A \cup B) \cap (A \cup C)$ = {1, 2, 3, 4} $\cap$ {1, 2, 3}

= {1, 2, 3} ..... (2)

From (1) and (2),

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Ex.6 List the sets A, B and C given that,

$A \cup B$ = {p, q, r, s}, $A \cup C$ = {q, r, s, t}, $A \cap B$ = {q, r}, $A \cap C$ = {q, s}.

Solution : Since $A \cap B$ = {q, r} and $A \cap C$ = {q, s}

q, r, s $\in$ A, q, r $\in$ B, q, s $\in$ C ..... (1)

Since $A \cup C$ = {q, r, s, t}, p $\notin$ A

Since p $\notin$ A and $A \cup B$ = {p, q, r, s} , p $\in$ B ..... (2)

Since $A \cup B$ = {p, q, r, s} , t $\notin$ A

Since t $\notin$ A and $A \cup C$ = {q, r, s, t}, t $\in$ C ..... (3)

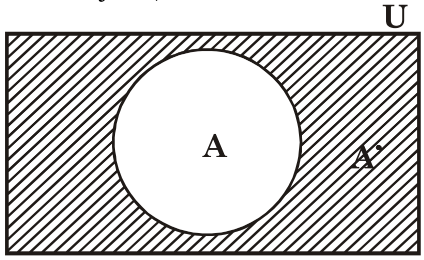
Hence from (1), (2) and (3) we have

A = {q, r, s} , B = {p, q, r}, C = {q, s, t}.

[3] Complement of a set :

"The complement of a set is the set of all those elements which do not belong to that set."

If U be the universal set and A $\in$ P(U) then the complement of set A is the set U – A and is denoted by A', A^C.



A'is the shaded part

A' = {x / x $\in$ U, x $\notin$ A}

If U = {a, b, c, d} and A = {a, c} then A' = {b, d}.

Properties of complementary set

(i) $A \cap A' = \phi$

(ii) $A \cup A' = U$

(iii) $U' = \phi$

(iv) $\phi' = U$

(v) $(A')' = A$

(vi) If $A \subset B$ then $B' \subset A'$

(vii) $(A \cap B) \cup (A \cap B') = A$

(viii) $(A \cap B)' = A' \cup B'$

(ix) $(A \cup B)' = A' \cap B'$.

Ex.7 If $U = \{x / 3 \le x \le 13, x \in N\}$, $A = \{y / 2 < y < 7, y \in N\}$, $B = \{3, 5, 7, 9\}$ then find (a) A' (b) $(B')'$ (c) $(A \cup B)'$.

Solution :

$U = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$

$A = \{3, 4, 5, 6\}$ $B = \{3, 5, 7, 9\}$

(a) $A' = \{7, 8, 9, 10, 11, 12, 13\}$

(b) $(B')' = B = \{3, 5, 7, 9\}$

(c) $A \cup B = \{3, 4, 5, 6\} \cup \{3, 5, 7, 9\}$

$A \cup B = \{3, 4, 5, 6, 7, 9\}$

$(A \cup B)' = \{8, 10, 11, 12, 13\}$

[4] De Morgan's Laws :

Theorem 4 Complement of a union is the intersection of complements,

i.e. $(A \cup B)' = A' \cap B'$.

Proof : Let x be any element of $(A \cup B)'$.

Then by definition of the complement,

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cap B')$

i.e. every member of $(A \cup B)'$ is also a member of $A' \cap B'$.

$\therefore (A \cup B)' \subseteq A' \cap B'$ ..... (1)

Let y be any element of $A' \cap B'$.

Then by definition of intersection,

$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

i.e. every member of $A' \cap B'$ is also a member of $(A \cup B)'$.

$\therefore A' \cap B' \subseteq (A \cup B)'$ ..... (2)

From (1) and (2) we conclude that

$(A \cup B)' = A' \cap B'$

Theorem 5 Complement of a intersection is the union of complements,

$(A \cap B)' = A' \cup B'$

Proof : Let x be any element of $(A \cap B)'$.

Then by definition of the complement,

$$\begin{aligned} x \in (A \cap B)' &\Rightarrow x \notin (A \cap B) \\ &\Rightarrow x \notin A \text{ and } x \notin B \\ &\Rightarrow x \in A' \text{ or } x \in B' \\ &\Rightarrow x \in (A' \cup B') \end{aligned}$$

i.e. every member of $(A \cap B)'$ is also a member of $A' \cup B'$.

$\therefore \ (A \cap B)' \subseteq A' \cup B'$ ..... (1)

Let y be any element of $A' \cup B'$.

Then by definition of intersection,

$$\begin{aligned} y \in A' \cup B' &\Rightarrow y \in A' \text{ or } y \in B' \\ &\Rightarrow y \notin A \text{ and } y \notin B \\ &\Rightarrow y \notin (A \cap B) \\ &\Rightarrow y \in (A \cap B)' \end{aligned}$$

i.e. every member of $A' \cup B'$ is also a member of $(A \cap B)'$.

$\therefore \ A' \cup B' \subseteq (A \cap B)'$ ..... (2)

From (1) and (2) we conclude that,

$(A \cup B)' = A' \cap B'$

Ex.8 If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 4, 6, 8} , B = {3, 6, 9} then prove that $(A \cap B)' = A' \cup B'$.

Solution :

$A \cap B = \{6\}$

$(A \cap B)' = \{1, 2, 3, 4, 5, 7, 8, 9\}$ ..... (1)

$A' = \{1, 3, 5, 7, 9\}$ and $B' = \{1, 2, 4, 5, 7, 8\}$

$A' \cup B' = \{1, 2, 3, 4, 5, 7, 8, 9\}$ ..... (2)

From (1) and (2)

$(A \cap B)' = A' \cup B'$

[5] Difference of Two sets :

"The difference of two sets A and B is the set of all those elements which belong to set A and not to B."

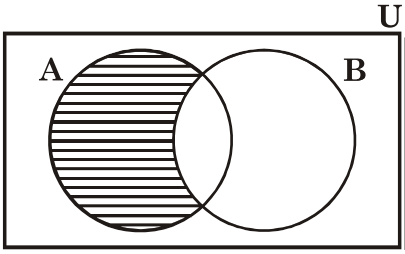
The difference of two sets A and B is denoted by A – B, i.e. 'A difference B'.

$A - B = \{x / x \in A \text{ and } x \notin B\}$

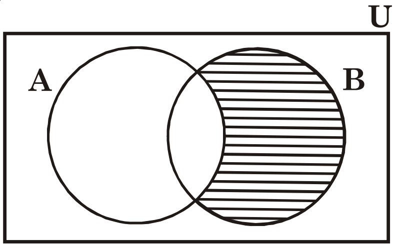
The difference of two set B and A is denoted by B – A, i.e. 'B difference A'.

$B - A = \{y \mid y \in B \text{ and } y \notin A\}$

The difference of two sets can be shown (shaded region) by Venn Diagram as under:



(A - B) is the shaded part



(B - A) is the shaded part

Properties of Difference sets

(i) $A - B \subset A$ and $B - A \subset B$

(ii) $A - B$, $A \cap B$, $B - A$ are mutually disjoint sets.

(iii) $A - (A - B) = A \cap B$ and $B - (B - A) = A \cap B$

(iv) If $A = B \Rightarrow (A - B) = \phi$

(v) $A - B = A \cap B'$

(vi) $A - \phi = A$

(vii) $U - A = A'$

(viii) $A \cup B = (A - B) \cup B$

Theorem 6 Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Solution :

Let $x \in A \cap (B - C) \Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and ($x \in B$ but $x \notin C$)

$\Rightarrow x \in (A \cap B)$ but $x \notin (A \cap C)$

$\Rightarrow x \in (A \cap B) - (A \cap C)$

$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ ..... (1)

Let $y \in (A \cap B) - (A \cap C) \Rightarrow y \in (A \cap B)$ but $y \notin A \cap C$

$\Rightarrow$ ($y \in A$ and $y \in B$) but $y \notin (A \cap C)$

$\Rightarrow$ ($y \in A$) and ($y \in B$ but $y \notin C$)

$\Rightarrow$ ($y \in A$) and $y \in (B - C)$

$\Rightarrow y \in A \cap (B - C)$

$\therefore (A \cap B) - (A \cap C) \subseteq A \cap (B - C)$ ..... (2)

From (1) and (2) we conclude that,

$$(A \cap B) - (A \cap C) = A \cap (B - C)$$

Ex.9 If $A = \{2, 3, 4\}$ and $B = \{4, 5, 6\}$ then find $A - B$, $B - A$.

Solution : $A - B = \{2, 3\}$, $B - A = \{5, 6\}$.

Ex.10 If $A = \{ x / 3 \le x \le 8, x \in N\}$, $B = \{ x / x^2 \le 4x, x \in N\}$ then verify that,

$A \cup B = (A - B) \cup B$.

Solution :

$A = \{ x / 3 \le x \le 8, x \in N\}$ $B = \{ x / x^2 \le 4x, x \in N\}$

$A = \{ 3, 4, 5, 6, 7, 8\}$ $B = \{1, 2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ ....(1)

$A - B = \{3, 4, 5, 6, 7, 8\} - \{1, 2, 3, 4\}$

$A - B = \{5, 6, 7, 8\}$

$(A - B) \cup B = \{5, 6, 7, 8\} \cup \{1, 2, 3, 4\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8\}$ ..... (2)

From (1) and (2) we conclude that,

$A \cup B = (A - B) \cup B$

Ex.11 If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$, $B = \{3, 6, 9\}$ then verify that,

$A - B \ne (B - A)'$.

Solution :

$A - B = \{2, 4, 6, 8\} - \{3, 6, 9\}$

$A - B = \{2, 4, 8\}$ ..... (1)

$B - A = \{3, 6, 9\} - \{2, 4, 6, 8\}$

$B - A = \{3, 9\}$

$(B - A)' = \{1, 2, 4, 5, 6, 7, 8\}$ ..... (2)

From (1) and (2) we conclude that,

$A - B \ne (B - A)'$.

Ex.12 If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$ and $C = \{4, 5, 6\}$ then show that,

(i) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution :

(i) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \{4, 5, 6\}$

L.H.S. = $(B - C) = \{2, 3\}$

$A \cap (B - C) = \{2, 3\}$ ..... (1)

R.H.S. = $(A \cap B) = \{2, 3, 4\}$

$(A \cap C) = \{4, 5\}$

$(A \cap B) - (A \cap C) = \{2, 3\}$ ..... (2)

From (1) and (2) we conclude that,

$A \cap (B - C) = (A \cap B) - (A \cap C)$.

(ii) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \{4, 5, 6\}$

L.H.S. = $(B \cap C) = \{4\}$

$A - (B \cap C) = \{1, 2, 3, 5\}$ ..... (1)

R.H.S. = $A - B = \{1, 5\}$

$A - C = \{1, 2, 3\}$

$(A - B) \cup (A - C) = \{1, 2, 3, 5\}$ ..... (2)

From (1) and (2) we conclude that,

$A - (B \cap C) = (A - B) \cup (A - C)$.

Ex.13 If $A = \{ x / x \leq 9, x \in N\}$,$B = \{ y / 3 \leq y \leq 7, y \in N$, odd number.$\}$, $C = \{ z / 1 \leq z \leq 7, z \in N$, even number$\}$ then prove that, $A - (B \cup C) = (A - B) \cap (A - C)$.

Solution :

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $B = \{3, 5, 7\}$, $C = \{2, 4, 6\}$

$B \cup C = \{2, 3, 4, 5, 6, 7\}$

$A - (B \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 3, 4, 5, 6, 7\}$

$A - (B \cup C) = \{1, 8, 9\}$ ..... (1)

$A - B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{3, 5, 7\}$

$= \{1, 2, 4, 6, 8, 9\}$

$A - C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 4, 6\}$

$= \{1, 3, 5, 7, 8, 9\}$

$(A - B) \cap (A - C) = \{1, 2, 4, 6, 8, 9\} \cap \{1, 3, 5, 7, 8, 9\}$

$(A - B) \cap (A - C) = \{1, 8, 9\}$ ..... (2)

From (1) and (2) we conclude that,

$A - (B \cup C) = (A - B) \cap (A - C)$.

[6] Symmetric Difference Set :

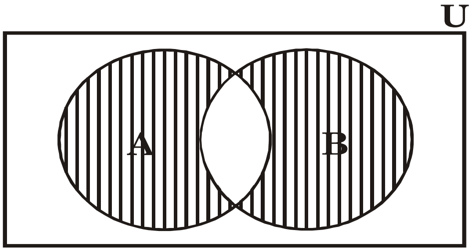
"A Difference set is called a symmetric difference of two sets if it contains all those elements which are in set A and not in set B or those which are in set B and not in set A."

"Symmetric difference set is the Union of two difference set A and B."

i.e. $A \Delta B = (A - B) \cup (B - A)$

$A \Delta B = \{x / x \in A$ or $x \in B$ but $x \notin A$ and $B\}$

The Symmetric difference of two sets can be shown (shaded region) by Venn Diagram as under :



Properties of symmetric difference set

(i) $A \Delta B = B \Delta A$
(ii) $A \Delta A = \phi$
(iii) $A \Delta (A \cap B) = A - B$
(iv) $(A \Delta B) \cup (A \cap B) = A \cup B$
(v) $A \Delta B = (A \cup B) - (A \cap B)$

Ex.14 A = {a, b, c}, B = {b, c, d} then find A Δ B.

Solution :

$A - B = \{a\}$
$B - A = \{d\}$
$A \Delta B = (A - B) \cup (B - A)$
$A \Delta B = \{a\} \cup \{d\} = \{a, d\}$

2.6 Order Pair

An ordered pair of objects consist of two elements x and y written in bracket (x, y) such that one of them, say, x is designated as first member and y as the second member.

For example:

(i) The natural number and their cubes can be represented by ordered pairs as,
(1, 1), (2, 8), (3, 27), ...

(ii) The points on the graph (2 dimension) can be represented by an ordered pair (x, y), where x is the first coordinate of x-axis and y is the second coordinate of y-axis. Thus, a point (x, y) represents an ordered pair.

(iii) Two ordered pair (a, b) and (x, y) will be equal if and only if, $a = x$ and $b = y$.

2.7 Cartesian Product

"If A and B be any two non-empty sets then the set of all ordered pair whose first member belong to set A and second member belong to set B is called the Cartesian product of A and B in that order."

It is denoted by A × B.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

Properties of Cartesian Product :

(i) If the number of elements in sets A and B are same but $A \times B \neq B \times A$.
(ii) $A \times B = B \times A$ if and only if $A = B$.
(iii) If number of elements in set A is p and in set B is q then the number of

elements in product set $A \times B$ are pq.

Theorem 7 In usual notation prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution :

Let (x, y) be any element of $A \times (B \cap C)$.

Then $(x, y) \in A \times (B \cap C) \Rightarrow x \in A$ and $y \in (B \cap C)$

$\Rightarrow x \in A$ and ($y \in B$ and $y \in C$)

$\Rightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

$\Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Rightarrow (x, y) \in (A \times B) \cap (A \times C)$

$\therefore A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. ..... (1)

Let (u, v) be any element of $(A \times B) \cap (A \times C)$.

$(u, v) \in (A \times B) \cap (A \times C) \Rightarrow (u, v) \in (A \times B)$ and $(u, v) \in (A \times C)$

$\Rightarrow$ ($u \in A$ and $v \in B$) and ($u \in A$ and $v \in C$)

$\Rightarrow u \in A$ and $v \in (B \cap C)$

$\Rightarrow (u, v) \in A \times (B \cap C)$

$\therefore (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. . .... (2)

From (1) and (2) by equality of sets, we have

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Ex.15 Let A = {1, 2, 3} and B = {a, b, c} then find $A \times B$ and $B \times A$.

Solution :

(i) $A \times B$ = {1, 2, 3} × {a, b, c}

= {(1, a), (1, b), (1, c) , (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)}

(ii) $B \times A$ = {a, b, c} × {1, 2, 3}

= {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)}

Ex.16 If A = {1, 2, 3} and B = {2, 3} prove that $A \times B \neq B \times A$.

Solution :

$A \times B$ = {1, 2, 3} × {2, 3}

$A \times B$ = {(1, 2) (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)}

$B \times A$ = {2, 3} × {1, 2, 3}

$B \times A$ = {(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

We notice that (1, 2) and (1, 3) which are the element of $A \times B$ are not elements of $B \times A$.

$\therefore A \times B \neq B \times A$

Ex.17 If A = {2, 3} then find A^2.

Solution :

$A^2 = A \times A$ = {2, 3} × {2, 3}

= {(2, 2), (2, 3), (3, 2), (3, 3)}

Ex.18 If A = {2, 3}, B = {6, 7}, C = {7, 9} then prove that,

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Solution :

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

$B \cup C = \{6, 7, 9\}$

$A \times (B \cup C) = \{2, 3\} \times \{6, 7, 9\}$

$A \times (B \cup C) = \{(2, 6), (2, 7), (2, 9), (3, 6), (3, 7), (3, 9)\}$ ..... (1)

$A \times B = \{2, 3\} \times \{6, 7\}$

$A \times B = \{(2, 6), (2,7), (3, 6), (3, 7)\}$

$A \times C = \{2, 3\} \times \{7, 9\}$

$A \times C = \{(2, 7), (2, 9), (3, 7), (3, 9)\}$

$(A \times B) \cup (A \times C) = \{(2, 6), (2, 7), (2, 9), (3, 6), (3, 7), (3, 9)\}$ ..... (2)

From (1) and (2),

$A \times (B \cup C) = (A \times B) \cup (A \times C)$

Ex.19 If $A = \{x / |x^2 - 1| < 10, x \in Z\}$, $B = \{x / |x - 1| < 2, x \in N\}$,

$C = \{x / |x| \leq 1, x \in Z\}$ then prove that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Solution :

$A = \{x / |x^2 - 1| \leq 10, x \in Z\}$

$\therefore A = \{-3, -2, -1, 0, 1, 2, 3\}$

$B = \{x / |x - 1| < 2, x \in N\}$

$\therefore B = \{1, 2\}$

$C = \{x / |x| \leq 1, x \in Z\}$

$\therefore C = \{-1, 0, 1\}$

L.H.S. = $B \cap C = \{1\}$

$A \times (B \cap C) = \{-3, -2, -1, 0, 1, 2, 3\} \times \{1\}$

$= \{(-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1), (3, 1)\}$ ..... (1)

R.H.S. = $A \times B = \{-3, -2, -1, 0, 1, 2, 3\} \times \{1, 2\}$

$A \times B = \{(-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1), (3, 1), (-3, 2), (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2), (3, 2)\}$

$A \times C = \{-3, -2, -1, 0, 1, 2, 3\} \times \{-1, 0, 1\}$

$= \{(-3, -1), (-2, -1), (-1, -1), (0, -1), (1, -1), (2, -1), (3, -1), (-3, 0), (-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0), (3, 0), (-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1), (3, 1)\}$

$(A \times B) \cap (A \times C) = \{(-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1), (3, 1)\}$ ... (2)

From (1) and (2),

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Ex.20 If $A = \{x \in Z / x^3 = x\}$, $B = \{x \in Z / x^2 - x = 0\}$ and $C = \{1, 2\}$ then prove that, $A \times (B - C) = (A \times B) - (A \times C)$.

Solution :

Let $A = \{x \in z / x^3 = x\}$

$x^3 = x \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0$

$\Rightarrow x(x-1)(x+1) = 0$

$\therefore x = 0$ or $x = 1$ or $x = -1$

$\therefore A = \{-1, 0, 1\}$

$B = \{x \in z / x^2 - x = 0\}$

$x^2 - x = 0 \Rightarrow x(x-1) = 0 \therefore x = 0$ or $x = 1$

$\therefore B = \{0, 1\}$

$\therefore A = \{-1, 0, 1\}, B = \{0, 1\}, C = \{1, 2\} \quad \therefore U = \{-1, 0, 1, 2\}$

$A' = \{2\}, B' = \{-1, 2\}, C' = \{-1, 0\}$

L.H.S. $= A \times (B - C)$

$B - C = \{0, 1\} - \{1, 2\} = \{0\}$

$A \times (B - C) = \{-1, 0, 1\} \times \{0\}$

$A \times (B - C) = \{(-1, 0), (0, 0), (1,0)\}$ ..... (1)

R.H.S. $= (A \times B) - (A \times C)$

$A \times B = \{-1, 0, 1\} \times \{0, 1\}$

$= \{(-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1)\}$

$A \times C = \{-1, 0, 1\} \times \{1, 2\}$

$= \{(-1, 1), (-1, 2), (0, 1), (0, 2), (1, 1), (1, 2)\}$

$(A \times B) - (A \times C) = \{(-1, 0), (0, 0), (1, 0)\}$ ..... (2)

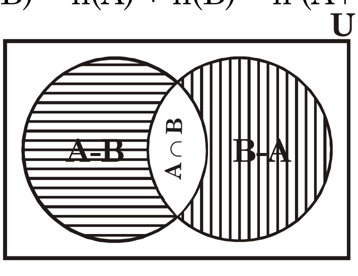
From (1) and (2),

$A \times (B - C) = (A \times B) - (A \times C)$

2.8 Practical Example

For any two non-empty finite set A and B, we have

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$



From the above diagram, we have

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$

If A and B are disjoint sets, then $A \cap B = \phi$ and $n(A \cap B) = 0$.

$\therefore n(A \cup B) = n(A) + n(B)$

If A, B, C is any 3 sets, then

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Ex.21 If A and B are two sets such that $n(A) = 17$, $n(B) = 23$, $n(A \cup B) = 38$ then find $n(A \cap B)$.

Solution :

Given, $n(A) = 17$, $n(B) = 23$, $n(A \cup B) = 38$, $n(A \cap B) = ?$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$38 = 17 + 23 - n(A \cap B)$

$n(A \cap B) = 17 + 23 - 38$

$n(A \cap B) = 2$

Ex.22 If $n(U) = 700$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 100$. Find $n(A' \cup B')$.

Solution :

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 200 + 300 - 100$

$= 400$

$n(A' \cup B') = n(A \cap B)'$

$= n(U) - n(A \cup B)$

$n(A' \cup B') = 700 - 400$

$= 300$

Ex.23 In a committee, 50 people speak Gujarati, 20 speak English and 10 speak both Gujarati and English. How many people speak at least one of these two languages?

Solution :

Let G be the set of people who speak Gujarati and

E be the set of people who speak English.

$\therefore$ $n(G) = 50$, $n(E) = 20$, $n(G \cap E) = 10$, we have to find $n(G \cup E)$.

Now, $n(G \cup E) = n(G) + n(E) - n(G \cap E)$

$n(G \cup E) = 50 + 20 - 10 = 60$

$\therefore$ 60 people speak at least one of these two languages.

:: EXERCISE : :

1. Define the following terms :

   (i) Finite and Infinite set (ii) Singleton set (iii) Empty set (iv) Equal set
   (v) Equivalent set (vi) Subsets (vii) Universal set (viii) Proper Subset
   (ix) Power set (x) Union of set (xi) Intersection of set (xii) Complement of set

(xiii) Difference set (xiv) De Morgan's Law (xv) Cartesian product of sets.

2. State and prove the distributive law of union over intersection.

3. Write properties of (i) Intersection set (ii) Union of set
(iii) Complement of set (iv) Difference set (v) Cartesian product of sets.

4. State the De Morgan's Law and prove by using examples.

5. Describe methods of representation of a set.

6. Write associative law of union and prove it.

7. Represent the following set in the set builder form.
A = {7, 8, 9, 10, 11}, B = {1, 4, 9, ...,100}

[Ans. A = {x/6<x<12, x∈N}, B = {x^2/0 < x ≤ 10, x∈N}]

8. What are the elements of set {x / x^2 = 9, 2x = 4, x ∈ R}. ***[Ans : {–3, 2, 3}]***

9. Are the following sets are equal ?
A = {x : x is a letter in the word REAP}
B = {x : x is a letter in the word PAPER}
C = {x : x is a letter in the word RAPE} ***[Ans : Yes]***

10. If A = {1, 2, 3, 4, 5}, find the total number of subsets of A ***[Ans. : 2^5 = 32]***

11. Are the sets ϕ, {0}, {ϕ} different ? ***[Ans. yes]***

12. Write down the singleton sets of A = {a, b, c}. ***[Ans : {a}, {b}, {c}]***

13. Write down the power set of A = {0}. ***[Ans : P (A) = {ϕ, {0}]***

14. How many cardinal number of the set of the letter in the word 'INDIA'.

[Ans : 4]

15. If A = {1, 2, 3, 4}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6} then find the value of

(i) (A ∪ B) ∩ C (ii) A ∪ (B ∩ C). ***[Ans : (i) {3, 4, 6} (ii) {1, 2, 3, 4, 6}]***

16. If A = {1, 3, 5, 7}, B = {–1, 0, 1} then find the value of (A ∪ B) – (A ∩ B).

[Ans. : {–1, 0, 3, 5, 7}]

17. Let U = {1, 2, 3 ... 9}, A = {1, 2, 3, 4}, B = {2, 4, 6, 8}, C = {3, 4, 5, 6} then find
(i) A' (ii) (A ∩ C)' (iii) (B – C). ***[Ans. (i) {5, 6, 7, 8, 9} (ii) {1, 2, 5, 6, 7, 8, 9} (iii) {2, 8}]***

18. Let U = {1, 2, 3, 4, 5, 6, 7}, A = {1, 2, 3, 4}, B = {2, 4, 6, 8} C = {3, 4, 5, 6} then
verify that, (i) A ∪ (B ∪ C) = {A ∪ B) ∪ C.
(ii) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).
(iii) (A ∩ B)' = A' ∪ B'.

(iv) $A \cap B' = (A' \cup B)'$.

(v) $A \cap (B' \cup C') = A \cap (B \cap C)'$.

(vi) $A - B = A \cap B' = B' - A'$.

(vii) $(A - B)' = A' \cup B$.

(viii) $(A \cup B) \cap B' = A - B$.

19. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 5\}$, $C = \{2, 4, 6, 8\}$ then verify that,
(i) $A - (A - B) = A \cap B$. (ii) $A \cap (B - C) = (A \cap B) - (B \cap C)$.

20. If $A = \{2, 3, 5, 6\}$, $B= \{3, 4, 6, 7\}$, $C = \{1, 2, 3, 4\}$ then verify that,
(i) $A - (B \cap C) = (A - B) \cup (A - C)$. (ii) $A - (B \cup C) = (A - B) \cap (A - C)$.

21. If $A = \{ a, b, c \}$, $B = \{ c, d \}$, $C = \{ a, e \}$ verify that,
(i) $A \times (B - C) = (A \times B) - (A \times C)$. (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
(iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

22. In a group of 26 persons, 8 take tea but not coffee and 16 take tea. How many take coffee but not tea ? ***[Ans. 10]***

24. In a village with a population of 5000, 3200 people are vegetarian, 2500 non-vegetarian and 1500 eat both. How many people in the village are vegetarians ? ***[Ans. 800]***

➢ **Select the appropriate answer from the given alternative answer.**

(M.C.Q.)

1. In .... set can be represented by showing all the members within bracket.
(a) Roster method (b) Union (c) Null (d) Rule method

2. In............, we list the property or properties satisfied by the element of set.
(a) Roster method (b) Union (c) Null (d) Rule method

3. Two sets A and B are said to be ...... if each element of A is an element of B and each element of B is also an element of A.
(a) Union (b) Equal (c) Null (d) Equivalent

4. The set A is the .... of set B if, each element of set A is an element of set B.
(a) Union (b) Equal (c) Subset (d) Equivalent

5. The family of all the .... of a given set is called the power set.
(a) Union (b) Equal (c) Subsets (d) Equivalent

6. The .... of two sets A and B is the set of all those elements which belong to set A and not to B

(a) Union **(b) Equal** **(c) Subsets** **(d) Difference**

7. $A \cap A' =$

(a) **A** **(b)** ϕ **(c) U** **(d)** $\{\phi\}$

8. $A \Delta B =$

(a) **A** **(b)** ϕ **(c)** $(A - B) \cup (B - A)$ **(d)** $(A \cup B)$

9. If A = {1, 2, 3, 4} and B = {4, 5, 6, 7} then A – B =

(a) **{1, 2, 3}** **(b) { 4 }** **(c)** **{1, 2, 3, 4, 5, 6, 7}** **(d)** ϕ

10. If A = {1, 2, 3, 4} and B = {4, 5, 6, 7} then $A \Delta B$ =

(a) **{1, 2, 3}** **(b) { 4 }** **(c)** **{1, 2, 3, 5, 6, 7}** **(d)** ϕ

11. If A = {1, 2, 3, 4, 5}, B = {2, 3, 4}, C = {4, 5, 6} then $A \cap (B - C)$ =

(a) **{1, 2, 3}** **(b) { 4 }** **(c)** **{1, 2, 3, 5, 6}** **(d)** **{2, 3}**

12. If A = {1, 2, 3, 4, 5}, B = {2, 3, 4}, C = {4, 5, 6} then $(A - B) \cup (A - C)$ =

(a) **{1, 2, 3}** **(b) {1, 2, 3, 5}** **(c)** **{4, 5, 6}** **(d)** **{2, 3}**

13. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 4, 6, 8}, B = {3, 6, 9} then $(A \cap B)$ =

(a) **{6}** **(b) {1, 2, 3, 5}** **(c)** **{3, 6, 9}** **(d)** **{2, 3}**

14. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9}, A = {2, 4, 6, 8}, B = {3, 6, 9} then A – B =

(a) **U** **(b)** **{1, 2, 3, 5}** **(c)** **{2, 4, 8}** **(d)** **{2, 3}**

15. $(A \cap B) - (A \cap C) =$

(a) **U** **(b)** ϕ **(c)** $(A - B) \cup (A - C)$ **(d)** $A \cap (B - C)$

:: ANS ::

1. **(a)** 2. **(d)** 3. **(b)** 4. **(c)** 5. **(c)** 6. **(d)** 7. **(b)** 8. **(c)**
9. **(a)** 10. **(c)** 11. **(d)** 12. **(b)** 13. **(a)** 14. **(c)** 15. **(d)**

UNIT-3 PERMUTATION AND COMBINATION

3.1 Introduction:

Permutations and combinations are used for counting and their major use is in getting decision in selection of the items from group of items with replacement and without replacement. Also, they are useful in probability theory concepts.

3.1.1 Fundamental Rules of Counting:

If one thing can be done in m different ways and second thing can be done in n different ways, then the two things together can be done in m x n different ways. e.g. If two coins are thrown simultaneously, the total outcomes will be four.

3.1.2 Kramp's Factorial Notation:

The product of the first n natural numbers 1, 2, 3, …, n is called n factorial and it is written as n!

Also, we can say n! = 1 x 2 x 3 x … x (n – 1) x n.

Note: 0! = 1

3.2 Meaning and definition of permutations:

3.2.1 Meaning:

A permutation is an arrangement or ordering of distinct objects where the order of the arrangement is important. For example, the words "cap", "acp" and "pac" are three different permutations of the same three letters.

3.2.2 Definition:

Each of the different arrangements made out of given distinct things by taking some or all of them at a time is called a permutation.
In general permutation is purely arrangement of number. Suppose r things have been taken from total n things (r ≤ n) then it is denoted by

$$n\text{Pr} = {}^{n}P_{r} = \frac{n!}{(n-r)!} = n.(n-1).(n-2).(n-3).....(n-r+1)$$

3.2.3 Properties:

Some properties of permutations are as follows

1) $^nP_r = n.(n-1).(n-2).(n-3).....(n-r+1)$

2) $^nP_r = \dfrac{n!}{(n-r)!}$

3) $^nP_n = n!$

4) $^nP_0 = 1$

3.3 Types of permutations:

3.3.1 Permutations of different things

Permutations of n different things taken r way where r ≤ n is denoted by ^nP_r.

Also, $^nP_r = n.(n-1).(n-2).(n-3)...(n-r+1)$

or $^nP_r = \dfrac{n!}{(n-r)!}$

3.3.2 Circular permutations

In the circular arrangement, the relative position of the other objects depends on the position of the object placed first. It is only then the arrangement of the remaining objects is made. Therefore, the circular permutation of n objects will be in (n – 1)! ways instead of n!

Also, if the arrangement of n different objects in such a way that no two similar things are close to each other than the number of ways will be $\frac{1}{2}(n-1)!$

3.3.3 Permutations of similar things

If the number of permutations of n things of which p things are of one kind, q things are of a second kind, r things are of a third kind and all the rest are different is given by

$$\frac{n!}{p!q!r!}$$

3.3.4 Restricted permutation

It can be considered in the following manner

i. The number of permutations of n different things taken r at a time in which p particular things do not occur is $^{n-p}P_r$

ii. The number of permutations of n different things taken r at a time in which p particular things are present is $^{n-p}P_{r-p}.{}^rP_p$

3.4 Illustrations:

1) In a bus, 9 seats are vacant. If 4 passengers enter into the bus, then in how many ways, they can sit?

Solution:

The total number of ways of arrangement of 4 passengers on 9 seats are $^{9}P_{4} = 9X8X7X6 = 3024$

2) How many five digits numbers can be formed by using the digits 1, 2, 3, 4, 5, 6? (i) How many of them are more than 50000? (ii) How many of them are multiple of 5?

Solution:

Here, we have to form five digits number by using 6 digits 1, 2, 3, 4, 5, 6. So, the total number of 5 digits numbers = $^{6}P_{5} = 6X5X4X3X2 = 720$

i. We know that if the first digit of the five number is 5 or more than five then the number is more than 50000. Therefore, from the given digits, the digits 5 and 6 can be placed at the first place in different $^{2}P_{1}$ways and then remaining five digits can be arranged in remaining four places in different $^{5}P_{4}$ ways.

   So, the total no. of ways of arrangement = $^{2}P_{1}X\,^{5}P_{4} = 240$.

   Hence 240 numbers are more than 50000.

ii. We know that if digits 0 or 5 is at the unit place then the number is multiple of 5. Now, from the given digits we can place the digits 5 at the unit place in $^{1}P_{1}$ way and remaining 5 digits can be arranged in remaining 4 places in different $^{5}P_{4}$ ways. So, total no. of ways of arrangement $= {}^{1}P_{1}X\,^{5}P_{4} = 120$. Hence, 120 numbers are multiple of 5.

3) The chief ministers of 18 states in India meet to discuss the problem of unemployment. In how many ways can they seat themselves at a round table if the Punjab and Bengal chief minister choose to sit together.

Solution:

Since the chief minister are sit at a round table, we shall have to fix the position of one of the chief ministers and then make the other 17 chief ministers take their seats. Since the Punjab and Bengal chief ministers are to sit together, consider them one. These 16 can now arranged among themselves in 16! ways. Further the Punjab and Bengal chief ministers can be arranged in 2! Ways.
Hence the required number of ways is 16! X 2!

4) Find the number of permutations of letters in the word ENGINEERING.

Solution:

Since the word ENGINEERING consists of 11 letters, in which there are 3 Es, 3 Ns, 2 Gs, 2 Is and 1 R, the total number of permutations is

$$\frac{11!}{3!.3!.2!.2!}$$

5) How many different words can be made out of the letters in the word ALLAHABAD. In how many of these will the vowels occupy the even places?

Solution:

The ALLAHABAD consists of 9 letters of which A is repeated four times, L is repeated twice and the rest all different. Hence the required number of the words are $\frac{9!}{4!.2!} = 7560$

Also, the word ALLAHABAD consists of 9 letters, there are 4 even places which can be filled up by the 4 vowels in 1 way only, since all the vowels are similar. Further the remaining 5 places can be filled up by the 5 consonants of which two are similar which can be filled in $\frac{5!}{2!}$ Ways. Hence required arrangements are $1X\frac{5!}{2!} = 60$

6) In how many different ways can 8 examination papers be arranged in a line so that the best and worst papers are never together?

Solution:

The total number of arrangements that can be made of 8 papers is 8!. Now let the best and the worst papers be taken together. These taken as one and the remaining 6 can be arranged amongst themselves in 7! Ways. In each of these arrangements the best and the worst paper can be arranged in 2! ways.

So, the total arrangement in which the best and the worst papers can come together are 7! X 2!

Hence, the number of arrangements in which the two particular papers are not together are 8! – 2 X 7! = 30240

7) If the letters of the word "WOMAN" be permuted and the word so formed be arranged as in a dictionary, what will be the rank of the word "WOMAN"?

Solution:

The total number of possible words will be $^5P_5 = 120$ since there are 5 alphabets.

The number of words beginning with A will be $^4P_4 = 24$. Thus, the number of words beginning with A, M, N and O are 4 X 24 = 96. The words beginning with W will have their ranks from 97 to 120.

The word beginning with W and having A, M and N in the second place are $3X\,^3P_3 = 18$.

So, the words beginning with W, O and A will be $^2P_2 = 2$

The words beginning with W and O will have their ranks from 97 + 18 = 115.

Now, the words beginning with W, O and M will have their ranks from 115 + 2 = 117 onwards.

So, we have WOMAN 117

WOMNA 118

Hence rank of the word "WOMAN" is 117.

3.5 Meaning and definition of combinations:

3.5.1 Meaning:

A combination is a way to select items from a set without considering the order of selection. For example, if you have three subjects: accountancy, statistics and mathematics, there are three combinations of two subjects you can choose: accountancy and statistics, statistics and mathematics or mathematics and accountancy.

3.5.2 Definition:

The number of ways of selecting r things out of n (n ≥ r) different things is known as combination.

In general, it is denoted by

$$_{n}C_{r} = \binom{n}{r} = {}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!.(n-r)!}$$

3.5.3 Properties:

Some properties of Combinations are as follows

1) $^{n}C_{n} = 1$
2) $^{n}C_{0} = 1$
3) $^{n}C_{r}.r! = {}^{n}P_{r}$
4) $^{n}C_{r} = {}^{n}C_{n-r}$
5) $^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

3.6 Types of combination

3.6.1 Combinations of things taken some or all at time

The total number of combinations of n different things taken some or all at a time is $2^{n} - 1$.

3.6.2 Some restricted combinations

It can be considered following ways:

i. The number of combinations of n things taken r at a time in which p particular things always occur is $^{n-p}C_{r-p}$.
ii. The number of combinations of n things taken r at a time in which p particular things never occur is $^{n-p}C_{r}$

3.7 Illustrations

1) In an examination in paper of Business Mathematics 10 questions are set. In how many different ways can an examinee choose 7 questions.

Solution:

The number of different choices is evidently equals to the number of ways in which 7 places can be filled up by 10 different things.

As we know $^{n}C_{r} = \dfrac{n!}{r!.(n-r)!}$

So, the required ways are $^{10}C_{7} = \dfrac{10!}{7!X\left(10-7\right)!} = \dfrac{10!}{7!X3!}$

$$= \frac{10X9X8X7!}{7!X1X2X3} = \frac{10X9X8}{1X2X3} = 120$$

2) In how many ways can 4 white and 3 black balls be selected from a box containing 20 white and 15 black balls.

Solution:

Here it is the case of selection of balls where 4 out of 20 white balls and 3 out of 15 black balls which can be considered as $^{20}C_{4}X\,^{15}C_{3}$

As we know $^{n}C_{r} = \dfrac{n!}{r!.(n-r)!}$

$$^{20}C_{4}X\,^{15}C_{3} = \frac{20!}{4!X\left(20-4\right)!}X\frac{15!}{3!X\left(15-3\right)!}$$

$$= \frac{20!}{4!X16!}X\frac{15!}{3!X12!} = 4845X455 = 2204475$$

3) In order to pass C.A. examination minimum marks have to be selected in each of the 7 subjects. In how many cases can a student fail?

Solution:

Each subject can be delt in two ways, the student may pass or fail in it. So, the 7 subjects can be dealt in 2^{7} ways. But this includes the case in which the student passes in all the 7 subjects. Excluding this, the number of ways in which the student can fail is $2^{7}-1=127$

3.8 Exercise

✓ Theoretical questions

1) What is permutation?
2) Give properties of permutations.
3) Explain different types of permutations.
4) What is Combination?
5) Give properties of combinations.
6) Explain different types of combinations.

✓ MCQs

1) The value of P (n, n-1) is
   a) n
   b) n!
   c) 2n
   d) 2n!

Answer: (b) n!

2) The number of ways in which 8 students can be seated in a line is
   a) 5040
   b) 50400
   c) 40230
   d) 40320

Answer: (d) 40320

3) If $^{n}P_{5} = 60^{n-1}P_{3}$, the value of n is
   a) 6
   b) 10
   c) 12
   d) 16

Answer: (b) 10

4) The number of squares that can be formed on a chessboard is
   a) 64
   b) 160
   c) 204
   d) 224

Answer: (c) 204

5) The number of ways 4 boys and 3 girls can be seated in a row so that they are alternate is
   a) 12
   b) 104
   c) 144
   d) 256

Answer: (c) 144

6) The number of ways 10 digit numbers can be written using the digits 1 and 2 is
   a) 2^{10}
   b) $^{10}C_2$
   c) 10!
   d) $^{10}C_1+^{9}C_2$

Answer: (a) 2^{10}

7) A coin is tossed n times, the number of all the possible outcomes is
   a) 2n
   b) 2^n
   c) C (n, 2)
   d) P (n, 2)

Answer: (b) 2^n

8) In how many ways 8 distinct toys can be distributed among 5 children?
   a) $^{8}P_{5}$
   b) $^{5}P_{8}$
   c) 5^{8}
   d) 8^{5}

Answer: (c) 5^{8}

9) There are 10 true-false questions in an examination. These questions can be answered in:
   a) 20 ways
   b) 100 ways
   c) 512 ways
   d) 1024 ways

Answer: (d) 1024 ways

10) In how many ways can we paint the six faces of a cube with six different colours?
   a) 30
   b) 6
   c) 6!
   d) None of the above

Answer: (a) 30

✓ Terminologies

$$^{n}P_{r} = n.(n-1).(n-2).(n-3)...(n-r+1)$$

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$_{n}C_{r} = \binom{n}{r} = {}^{n}C_{r} = \frac{^{n}P_{r}}{r!} = \frac{n!}{r!.(n-r)!}$$

✓ Short questions

1) If $^{n}C_{3} = {}^{n}C_{5}$, then find the value of $^{2n}C_{2}$

Answer: 120

2) If $^{10}P_{r} = 604800$ and $^{10}C_{r} = 120$ then show that r = 7.

✓ Practical examples

1) There are five routes for journey from destination A to Destination B. In how many different ways can a person go from A to B and return, if for returning
   i. Any one of the routes is taken
   ii. The same route is taken
   iii. The same route is not taken

   Answer: 25, 5, 20

2) How many telephone connections can be allotted with 5 and 6 digits from the natural numbers 1 to 9 inclusive?

   Answer: 59049, 531441

3) In how many ways can a chairman and a vice – chairman of a board of 6 members can occupy their seats?

   Answer: 30

4) In how many different ways, 3 rings of a lock can combine when each ring has 10 digits 0 to 9? If the lock opens in only one combination of 3 digits how many unsuccessful events are possible?

   Answer: 999

5) Find how many four letter words can be formed out of the word LOGARITHMS. (the words may not have any meanings)

   Answer: 5040

6) Indicate how many 4 digits numbers greater than 7000 can be formed from the digits 3, 5, 7, 8, 9.

   Answer: 72

7) In how many ways can 5 Accountancy book, 3 Statistics book and 3 Mathematics book be arranged if the books of each different subjects are kept together.

   Answer: 25920

8) In how many ways can 5 boys and 5 girls be seated around a table so that no 2 boys are adjacent.

   Answer: 2880

9) In how many ways can 4 Indians and 4 Americans be seated at a round table so that no two Indians may be together?

   Answer: 144

10) Find the number of permutations of the word ACCOUNTANT

Answer: 226800

11) The letter of the word ZENITH are written in all possible orders. How many words are possible if all these words are written out as in a dictionary? What is the rank of the word ZENITH?

    Answer: 616

12) Find the numbers less than 1000 and divisible by 5 which can be formed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 such that each digit does not occur more than once in each number.

    Answer:154

13) Six papers are set in an examination, of which two are 'Statistics' In how many different orders can the papers be arranged so that the two statistics papers are not together?

Answer: 480

14) A cricket team is to be formed consisting of 2 wicket keeper, 4 bowlers and 5 batsmen from a group of players containing 4 wicket keeper, 8 bowler and 11 batsmen. Find the number of ways a cricket team can be constituted.

Answer: ${}^{4}C_{2} X\, {}^{8}C_{4} X\, {}^{11}C_{5}$

15) In how many ways can you choose six out of nine questions? In how many of these ways the first question is always excluded? In how many ways the first and second questions are always included?

Answer: 84, 28, 35

16) Out of 17 consonants and 5 vowels, how many different words can be formed each consisting of 3 consonants and 2 vowels?

Answer: ${}^{17}C_{3} X\, {}^{5}C_{2} X\, {}^{5}P_{5} = 816000$

UNIT-4 BINOMIAL THEOREM

4.1 Introduction:

As we can carry some expansions directly as follows

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$
$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

But it is tough when power is high in that case Binomial Theorem is very useful to get expansion.

4.2 Binomial Theorem:

- **Statement:** For given polynomial $(x+a)^n$ where n ϵ N can be expand as follows

$$\left(x+a\right)^n = {}^nc_0x^na^0 + {}^nc_1x^{n-1}a^1 + {}^nc_2x^{n-2}a^2 + {}^nc_3x^{n-3}a^3 + \ldots + {}^nc_nx^{n-n}a^n$$

Note: If the expression has negative sigh then the same theorem can be used by making some corrections in the above statement.

$$\left(x-a\right)^n = \left(x+(-a)\right)^n = {}^nc_0x^n(-a)^0 + {}^nc_1x^{n-1}(-a)^1 + {}^nc_2x^{n-2}(-a)^2 + \ldots +$$
$${}^nc_nx^{n-n}(-a)^n$$

4.3 Characteristics of Binomial Theorem:

Some characteristics are as follows

1) Total number of terms in the expansion of $(x+a)^n$ is n + 1.
2) The coefficients of the different terms in the expansion are ${}^nC_{0,}\ {}^nC_{1,}\ {}^nC_{2,}\ {}^nC_{3,}\ldots{}^nC_n$
3) Sum of power of x and a in any term is always equals to n.
4) In every term power of the first item is decreased by one and power of the second item is increased by 1.
5) The coefficients of the every term equidistance from the middle term on both sides are equal.
6) If power is even, then expanding terms are odd with unique middle term.

7) If power is odd then expanding terms are even with two middle terms.

4.4 Uses of Binomial Theorem

The major uses of the Binomial Expansion Theorem are as follows:

1) To get expansion of the form of $(x+a)^n$ where n ϵ N.
2) To get middle term/s of $(x+a)^n$ type polynomial.
3) To get coefficient of the specific term of $(x+a)^n$ type polynomial.
4) To get constant term/s of $(x+a)^n$ type polynomial.
5) To get approximate value.

4.5 Binomial Coefficient (Pascal Triangle):

- **Introduction:**

Coefficient of Binomial Expansion also can be found out by Pascal Triangle as follows.

Power n	Coefficients	Sum of coefficients
1	1 1	$2 = 2^1$
2	1 2 1	$4 = 2^2$
3	1 3 3 1	$8 = 2^3$
4	1 4 6 4 1	$16 = 2^4$
5	1 5 10 10 5 1	$32 = 2^5$
6	1 6 15 20 15 6 1	$64 = 2^6$
$\vdots$	$\cdots$	$\vdots$
n	$+ \; - \; + \; - \; + \; - \; + \; -$	2^n

4.5.1 Properties of Pascal Triangle

Some properties are as follows

1) Both the sides of the triangles are 1.
2) Succeeding lines coefficients can be find out by adding previous line numbers.
3) If sign of the expression is positive then all terms will be positive sign.
4) If sign of the expression is negative then first term will have positive sign then second term will have negative sign and so on.

4.5.2 Uses of Pascal Triangle

The major uses of the Pascal Triangle are as follows:

1) To get coefficient for the expansion.
2) To get sum of the coefficients.

4.6 Expansion of Binomial Position of Terms and Middle Terms

- **General term of Binomial Expansion Theorem for** $(x+a)^n$.

From the statement of the Binomial Expansion Theorem we can said that

$$(x+a)^n = {}^nc_0 x^n a^0 + {}^nc_1 x^{n-1} a^1 + {}^nc_2 x^{n-2} a^2 + {}^nc_3 x^{n-3} a^3 + \ldots + {}^nc_n x^{n-n} a^n$$

Here power of the expression is n so total number of expanding terms are (n+1). Suppose the first term of the expansion is T1, second term is T2 likewise then it can be considered as follows.

First Term = $T_1 = T_{0+1} = {}^nc_0 x^{n-0} a^0$

Second Term = $T_2 = T_{1+1} = {}^nc_1 x^{n-1} a^1$

Third Term = $T_3 = T_{2+1} = {}^nc_2 x^{n-2} a^2$

………………………………………

(n + 1)th Term = $T_{n+1} = {}^nc_n x^{n-n} a^n$

So, the general form of the Binomial Theorem can be considered as

$$T_{r+1} = {}^nc_r x^{n-r} a^r$$

Note: Middle terms in Binomial Expansion can be find out as follows where n is the power of the given expression:

1) If the value of the n is even then middle term will be $\left(\frac{n}{2}+1\right)$

2) If the value of the n is odd then middle terms will be $\left(\frac{n+1}{2}, \frac{n+1}{2}+1\right)$

4.7 Illustrations

1) Expand $\left(\frac{x}{3}+\frac{2}{y}\right)^4$

Solution:

$$\left(\frac{x}{3}+\frac{2}{y}\right)^4 = {}^4C_0\left(\frac{x}{3}\right)^4\left(\frac{2}{y}\right)^0 + {}^4C_1\left(\frac{x}{3}\right)^3\left(\frac{2}{y}\right)^1 + {}^4C_2\left(\frac{x}{3}\right)^2\left(\frac{2}{y}\right)^2 +$$

$$+{}^4C_3\left(\frac{x}{3}\right)^1\left(\frac{2}{y}\right)^3 + {}^4C_4\left(\frac{x}{3}\right)^0\left(\frac{2}{y}\right)^4$$

$$= \frac{x^4}{81} + \frac{8x^3}{27y} + \frac{8x^2}{3y^2} + \frac{32x}{3y^3} + \frac{16}{y^4}$$

2) Expand $\left(\frac{3x}{4}-\frac{4}{3x}\right)^5$

Solution:

$$\left(\frac{3x}{4}-\frac{4}{3x}\right)^5=\left(\frac{3x}{4}+\left(-\frac{4}{3x}\right)\right)^5={}^5C_0\left(\frac{3x}{4}\right)^5\left(\frac{-4}{3x}\right)^0+{}^5C_1\left(\frac{3x}{4}\right)^4\left(\frac{-4}{3x}\right)^1+$$

$$+{}^5C_2\left(\frac{3x}{4}\right)^3\left(\frac{-4}{3x}\right)^2+{}^5C_3\left(\frac{3x}{4}\right)^2\left(\frac{-4}{3x}\right)^3+{}^5C_4\left(\frac{3x}{4}\right)^1\left(\frac{-4}{3x}\right)^4+{}^5C_5\left(\frac{3x}{4}\right)^0\left(\frac{-4}{3x}\right)^5$$

$$=\frac{243x^5}{1024}-\frac{135x^3}{64}+\frac{15x}{2}-\frac{40}{3x}+\frac{320}{27x^3}-\frac{1024}{243x^5}$$

3) Find value of $\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5$

Solution:

$$\left(\sqrt{2}+1\right)^5={}^5C_0\left(\sqrt{2}\right)^5(1)^0+{}^5C_1\left(\sqrt{2}\right)^4(1)^1+{}^5C_2\left(\sqrt{2}\right)^3(1)^2+$$

$$+{}^5C_3\left(\sqrt{2}\right)^2(1)^3+{}^5C_4\left(\sqrt{2}\right)^1(1)^4+{}^5C_5\left(\sqrt{2}\right)^0(1)^5$$

……………………………………………… (1)

$$\left(\sqrt{2}-1\right)^5=\left(\sqrt{2}+(-1)\right)^5={}^5C_0\left(\sqrt{2}\right)^5(-1)^0+{}^5C_1\left(\sqrt{2}\right)^4(-1)^1+$$

$$+{}^5C_2\left(\sqrt{2}\right)^3(-1)^2+{}^5C_3\left(\sqrt{2}\right)^2(-1)^3+{}^5C_4\left(\sqrt{2}\right)^1(-1)^4+{}^5C_5\left(\sqrt{2}\right)^0(-1)^5$$

……………………………………………………… (2)

By subtracting (1) and (2) and further simplifying we get

$$\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5=2\left({}^5C_1\left(\sqrt{2}\right)^4(1)^1\right)+2\left({}^5C_3\left(\sqrt{2}\right)^2(1)^3\right)+$$

$$2\left({}^5C_5\left(\sqrt{2}\right)^0(1)^5\right)$$

$$\left(\sqrt{2}+1\right)^5-\left(\sqrt{2}-1\right)^5=82$$

4) Expand $\left(y+\frac{1}{10y}\right)^8$ Simplify, each term as far as possible. Use your expansion to evaluate $(1.1)^8$ correct to four places of decimal.

Solution:

$$\left(y+\frac{1}{10y}\right)^8 = {}^8C_0(y)^8\left(\frac{1}{10y}\right)^0 + {}^8C_1(y)^7\left(\frac{1}{10y}\right)^1 + {}^8C_2(y)^6\left(\frac{1}{10y}\right)^2 +$$

$$+{}^8C_3(y)^5\left(\frac{1}{10y}\right)^3 + {}^8C_4(y)^4\left(\frac{1}{10y}\right)^4 + {}^8C_5(y)^3\left(\frac{1}{10y}\right)^5 + {}^8C_6(y)^2\left(\frac{1}{10y}\right)^6 +$$

$$+{}^8C_7(y)^1\left(\frac{1}{10y}\right)^7 + {}^8C_8(y)^0\left(\frac{1}{10y}\right)^8$$

…………………………………………………………… (1)

Now,

$$(1.1)^8 = (1+0.1)^8 = \left(1+\frac{1}{10}\right)^8$$

…………………………………………… (2)

By Comparing (1) and (2) we can get y = 1. Now take y = 1 in equation (1) so we can

get value of $(1.1)^8$ which is 2.14358881

5) Find the 5th term in the expansion of $\left(\frac{3x}{4}+\frac{4}{3x}\right)^{12}$

Solution:

Compare given expression with $(x+a)^n$

So, $n=12, x=\frac{3x}{4}, a=\frac{4}{3x}, T_n = 5^{th}\,Term$, $r=4$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{4+1} = {}^{12}C_4\left(\frac{3x}{4}\right)^{12-4}\left(\frac{4}{3x}\right)^4$$

$$T_5 = \frac{40095.x^4}{256}$$

6) Find the 3rd term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^8$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 8, x = \frac{4x}{5}, a = \frac{-5}{2x}, T_n = 3^{rd} Term$, $r = 2$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{2+1} = {}^8C_2 \left(\frac{4x}{5}\right)^{8-2} \left(\frac{-5}{2x}\right)^2$$

$$T_3 = \frac{28672.x^4}{625}$$

7) Find the nth term in the expansion of $\left(x - \frac{1}{x^2}\right)^{3n}$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 3n, x = x, a = \frac{-1}{x^2}, T_n = n^{th} Term$, $r = n-1$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{(n-1)+1} = {}^{3n}C_{n-1} (x)^{3n-(n-1)} \left(\frac{-1}{x^2}\right)^{n-1}$$

$$T_n = (-1)^{n-1}.\frac{(3n)!x^3}{(n-1)!(2n+1)!}$$

8) Find the middle term in the expansion of $\left(\frac{a}{x} + bx\right)^{12}$

Solution:

Compare given expression with $(x+a)^n$

So,

$$n = 12, x = \frac{a}{x}, a = bx, Middle\_Term = 7^{th} Term = T_7,\ r = 6$$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{6+1} = {}^{12}C_6 \left(\frac{a}{x}\right)^{12-6} (bx)^6$$

$$T_7 = 924.a^6.b^6$$

9) Find the middle term in the expansion of $\left(\frac{2x}{3} - \frac{3y}{2}\right)^9$

Solution:

Compare given expression with $(x+a)^n$

So,

$$n = 9, x = \frac{2x}{3}, a = \frac{-3y}{2}, Middle\_Terms\_are,$$

$5^{th} Term = T_5,\ r = 4$	$6^{th} Term = T_6,\ r = 5$
$T_{r+1} = {}^nC_r x^{n-r} a^r$	$T_{r+1} = {}^nC_r x^{n-r} a^r$
$T_{4+1} = {}^9C_4 \left(\frac{2x}{3}\right)^{9-4} \left(\frac{-3y}{2}\right)^4$	$T_{5+1} = {}^9C_5 \left(\frac{2x}{3}\right)^{9-5} \left(\frac{-3y}{2}\right)^5$
$T_5 = 84.x^5 y^4$	$T_6 = -189.x^4 y^5$

10) Find the middle term in the expansion of $(1+x)^{2n}$

Solution:

Compare given expression with $(x+a)^n$

So,

$$n = 2n, x = 1, a = x, Middle\_Term = (n+1)^{th} Term = T_{n+1},\ r = n$$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{n+1} = {}^{2n}C_n (1)^{2n-n} (x)^n$$

$$T_{n+1} = \frac{(2n)!}{(n!)^2}.x^n$$

11) If k is a real number and if the middle term in the expansion of $\left(\frac{k}{2} + 2\right)^8$ is 1120, then find value of k.

Solution:

Compare given expression with $(x+a)^n$

So,

$$n=8, x=\frac{k}{2}, a=2, Middle\_Term=5^{th}Term=T_5,\ r=4$$

$$T_{r+1}={}^nC_r x^{n-r}a^r$$

$$T_{4+1}={}^8C_4\left(\frac{k}{2}\right)^{8-4}(2)^4$$

$$T_5=70.\left(\frac{k}{2}\right)^4 2^4=\frac{70.k^4}{2^4}.2^4=70.k^4$$ ……………………….. (1)

But middle term is 1120 is given. So, from equation (1)

$$T_5=1120=70.k^4$$

$$\therefore k^4=16 \Rightarrow k=\pm 2$$

12) Find the coefficient of x^{32} and x^{-17} in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$ and show that their sum is zero.

Solution:

Compare given expression with $(x+a)^n$

So, $n=15, x=x^4, a=\frac{-1}{x^3}, r=r$

$$T_{r+1}={}^nC_r x^{n-r}a^r$$

$$T_{r+1}={}^{15}C_r\left(x^4\right)^{15-r}\left(\frac{-1}{x^3}\right)^r$$

$$T_{r+1}={}^{15}C_r.x^{60-4r}.\frac{(-1)^r}{x^{3r}}$$

$$T_{r+1}=(-1)^r.{}^{15}C_r.x^{60-7r}$$

……………………………………………….. (1)

Compare equation (1) with x^{32} and x^{-17} respectively.

$x^{32}=x^{60-7r}$	$x^{-17}=x^{60-7r}$
$\therefore 32=60-7r$	$\therefore -17=60-7r$
$\therefore r=4$	$\therefore r=11$
Put $r=4$ in equation (1)	Put $r=11$ in equation (1)
$\therefore T_5=(-1)^4.{}^{15}C_4.x^{60-7(4)}$	$\therefore T_{12}=(-1)^{11}.{}^{15}C_{11}.x^{60-7(11)}$

$\therefore T_5 = {}^{15}C_4.x^{32}$ ……………. (2)	$\therefore T_{12} = -{}^{15}C_{11}.x^{-17}$ ……………. (3)

From equation (2) and (3)

$$\therefore {}^{15}C_4 + \left(-{}^{15}C_{11}\right) = 0$$

So, we can say that sum of the coefficient of x^{32} and x^{-17} in the given expansion is 0.

13) Find the coefficient of x^{-2} in the expansion of $\left(2x - \frac{1}{x^2\sqrt{3}}\right)^{10}$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 10, x = 2x, a = \frac{-1}{x^2\sqrt{3}}, r = r$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{r+1} = {}^{10}C_r (2x)^{10-r}\left(\frac{-1}{x^2\sqrt{3}}\right)^r$$

$$T_{r+1} = {}^{10}C_r 2^{10-r} x^{10-r} \frac{(-1)^r}{x^{2r}\left(\sqrt{3}\right)^r}$$

$$T_{r+1} = (-1)^r . {}^{10}C_r . \frac{2^{10-r}}{\left(\sqrt{3}\right)^r} . x^{10-3r} \quad \text{………………………………………….. (1)}$$

Compare equation (1) with x^{-2}.

$$x^{-2} = x^{10-3r}$$

$$\therefore -2 = 10 - 3r$$

$$\therefore r = 4$$

Put $r = 4$ in equation (1)

$$T_5 = (-1)^4 . {}^{10}C_4 . \frac{2^{10-4}}{\left(\sqrt{3}\right)^4} . x^{10-3(4)}$$

$$T_5 = \frac{4480}{3} . x^{-2}$$

So, the coefficient of x^{-2} in the given expansion is $\frac{4480}{3}$

14) Find the term independent of x in $\left(2x + \frac{1}{3x^2}\right)^9$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 9, x = 2x, a = \frac{1}{3x^2}, r = r$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{r+1} = {}^9C_r (2x)^{9-r} \left(\frac{1}{3x^2}\right)^r$$

$$T_{r+1} = {}^9C_r 2^{9-r} x^{9-r} \frac{1^r}{3^r x^{2r}}$$

$$T_{r+1} = {}^9C_r \frac{2^{9-r}}{3^r} x^{9-3r}$$

………………………………………….. (1)

Compare equation (1) with x^0.

$$x^0 = x^{9-3r}$$

$$\therefore 0 = 9 - 3r$$

$$\therefore r = 3$$

Put $r = 3$ in equation (1)

$$T_{3+1} = {}^9C_3 \frac{2^{9-3}}{3^3} x^{9-3(3)}$$

$$T_4 = (84) . \frac{64}{27} x^0$$

$$T_4 = \frac{1792}{9}$$

So, the term independent of x in the given expansion is $\frac{1792}{9}$

15) Find the term independent of x (constant term) in $\left(\frac{4x^2}{3} - \frac{3}{2x}\right)^9$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 9, x = \frac{4x^2}{3}, a = \frac{-3}{2x}, r = r$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{r+1} = {}^9C_r \left(\frac{4x^2}{3}\right)^{9-r} \left(\frac{-3}{2x}\right)^r$$

$$T_{r+1} = {}^9C_r \left(\frac{4^{9-r} x^{2(9-r)}}{3^{9-r}}\right)\left(\frac{(-1)^r 3^r}{2^r x^r}\right)$$

$$T_{r+1} = (-1)^r . {}^9C_r \left(\frac{4^{9-r} 3^r}{2^r . 3^{9-r}}\right)\left(\frac{x^{18-2r}}{x^r}\right)$$

$$T_{r+1} = (-1)^r . {}^9C_r \left(\frac{4^{9-r} 3^r}{2^r . 3^{9-r}}\right) x^{18-3r}$$ ………………………………….. (1)

Compare equation (1) with x^0.

$$x^0 = x^{18-3r}$$

$$\therefore 0 = 18 - 3r$$

$$\therefore r = 6$$

Put $r = 6$ in equation (1)

$$T_{6+1} = (-1)^6 . {}^9C_6 \left(\frac{4^{9-6} 3^6}{2^6 . 3^{9-6}}\right) x^{18-3(6)}$$

$$T_7 = (84).(27) x^0$$

$$T_7 = 2268$$

So, the term independent of x in the given expansion is $T_7 = 2268$

16) Show that there is no term independent of x in the expansion of $\left(2x^2 - \dfrac{1}{4x}\right)^{11}$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 11, x = 2x^2, a = \dfrac{-1}{4x}, r = r$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{r+1} = {}^{11}C_r \left(2x^2\right)^{11-r} \left(\frac{-1}{4x}\right)^r$$

$$T_{r+1} = {}^{11}C_r \left(2^{11-r} x^{2(11-r)}\right)\left(\frac{(-1)^r}{4^r x^r}\right)$$

$$T_{r+1} = (-1)^r . {}^{11}C_r \left(\frac{2^{11-r}}{4^r}\right)\left(\frac{x^{22-2r}}{x^r}\right)$$

$$T_{r+1} = (-1)^r . {}^{11}C_r \left(\frac{2^{11-r}}{\left(2^2\right)^r}\right) . x^{22-3r}$$

$$T_{r+1} = (-1)^r . {}^{11}C_r . 2^{11-3r} . x^{22-3r}$$

……………………………….. (1)

Compare equation (1) with x^0.

$$x^0 = x^{22-3r}$$

$$\therefore 0 = 22 - 3r$$

$\therefore r = 22/3$ This is not positive integer. So, there is no existence of the constant term in the given expansion.

17) Show that the term independent of x in the expansion of $\left(x + \dfrac{1}{x}\right)^{2n}$ is

$$\frac{1.3.5.....(2n-1)}{n!} . 2^n$$

Solution:

Compare given expression with $(x+a)^n$

So, $n = 2n, x = x, a = \frac{1}{x}, r = r$

$$T_{r+1} = {}^{n}C_r x^{n-r} a^r$$

$$T_{r+1} = {}^{2n}C_r (x)^{2n-r} \left(\frac{1}{x}\right)^r$$

$$T_{r+1} = {}^{2n}C_r x^{2n-r} \left(\frac{1^r}{x^r}\right)$$

$$T_{r+1} = {}^{2n}C_r x^{2n-2r}$$

………………………………………………….. (1)

Compare equation (1) with x^0.

$$x^0 = x^{2n-2r}$$

$$\therefore 0 = 2n - 2r$$

$$\therefore r = n$$

Put $r = n$ in equation (1)

$$T_{n+1} = {}^{2n}C_n x^{2n-2n}$$

$$T_{n+1} = {}^{2n}C_n x^0$$

$$T_{n+1} = {}^{2n}C_n = \frac{(2n)!}{n!.(2n-n)!} = \frac{(2n)!}{n!.n!}$$

$$T_{n+1} = \frac{(2n).(2n-1).(2n-2).(2n-3).(2n-4).(2n-5).....4.3.2.1}{n!.n!}$$

$$T_{n+1} = \frac{\left[(2n).(2n-2).(2n-4).....4.2\right].\left[(2n-1).(2n-3).(2n-5).....3.1\right]}{n!.n!}$$

$$T_{n+1}=\frac{[2.2.2.....2.2].[(n).(n-1).(n-2).....2.1].[(2n-1).(2n-3).(2n-5).....3.1]}{n!.n!}$$

$$T_{n+1}=\frac{2^n.[n.(n-1).(n-2).....2.1].[(2n-1).(2n-3).(2n-5).....3.1]}{n!.n!}$$

$$T_{n+1}=\frac{2^n.n!.[(2n-1).(2n-3).(2n-5).....3.1]}{n!.n!}$$

$$T_{n+1}=\frac{2^n.[(2n-1).(2n-3).(2n-5).....3.1]}{n!}$$

So, the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is

$$\frac{1.3.5.....(2n-1)}{n!}.2^n$$

18) If the coefficient of x^2 and x^3 in the expansion of $(3+kx)^9$ are equal, find the value of k.

Solution:

Compare given expression with $(x+a)^n$

So, $n=9, x=3, a=kx, r=r$

$$T_{r+1}={}^nC_r x^{n-r}a^r$$

$$T_{r+1}={}^9C_r(3)^{9-r}(kx)^r$$

$$T_{r+1}={}^9C_r(3)^{9-r}k^r x^r$$

………………………………………………….. (1)

Compare equation (1) with x^2 and x^3 respectively

$x^2=x^r$	$x^3=x^r$
$\therefore 2=r$	$\therefore 3=r$
$\therefore r=2$	$\therefore r=3$
Put $r=2$ in equation (1)	Put $r=3$ in equation (1)
$T_{2+1}={}^9C_2(3)^{9-2}k^2x^2$	$T_{3+1}={}^9C_3(3)^{9-3}k^3x^3$
$T_3={}^9C_2(3)^7k^2x^2$ ……………. (2)	$T_4={}^9C_3(3)^6k^3x^3$ ……………. (3)

The coefficient of x^2 and x^3 in the expansion of $\left(3+kx\right)^9$ are equal. So, from equation (2) and (3)

$$^9C_2\left(3\right)^7k^2={}^9C_3\left(3\right)^6k^3$$

$$\therefore k=\frac{^9C_2\left(3\right)^7}{^9C_3\left(3\right)^6}=\frac{36(3)}{84}=\frac{9}{7}$$

19) If the coefficient of x^7 and x^8 in the expansion of $\left(3+\frac{x}{2}\right)^n$ are equal, find the value of n.

Solution:

Compare given expression with $\left(x+a\right)^n$

So, $n=n,x=3,a=\frac{x}{2},r=r$

$$T_{r+1}={}^nC_rx^{n-r}a^r$$

$$T_{r+1}={}^nC_r\left(3\right)^{n-r}\left(\frac{x}{2}\right)^r$$

$$T_{r+1}={}^nC_r\frac{\left(3\right)^{n-r}}{2^r}.x^r$$

……………………………………………….. (1)

Compare equation (1) with x^7 and x^8 respectively

$x^7=x^r$	$x^8=x^r$
$\therefore 7=r$	$\therefore 8=r$
$\therefore r=7$	$\therefore r=8$
Put $r=7$ in equation (1)	Put $r=8$ in equation (1)
$T_{7+1}={}^nC_7\frac{\left(3\right)^{n-7}}{2^7}.x^7$	$T_{8+1}={}^nC_8\frac{\left(3\right)^{n-8}}{2^8}.x^8$
$T_8={}^nC_7\frac{\left(3\right)^{n-7}}{2^7}.x^7$ ………… (2)	$T_9={}^nC_8\frac{\left(3\right)^{n-8}}{2^8}.x^8$ ……………. (3)

The coefficient of x^7 and x^8 in the expansion of $\left(3+\frac{x}{2}\right)^n$ are equal. So, from equation (2) and (3)

$$ {}^{n}C_7 \frac{(3)^{n-7}}{2^7} = {}^{n}C_8 \frac{(3)^{n-8}}{2^8} $$

$$ \frac{(3)^{n-7} 2^8}{2^7 (3)^{n-8}} = \frac{{}^{n}C_8}{{}^{n}C_7} \Rightarrow (3)^{n-7-(n-8)} 2^{8-7} = \frac{{}^{n}C_8}{{}^{n}C_7} $$

$$ \therefore (3)^{n-7-n+8} 2^1 = \frac{n!}{8!(n-8)!} \frac{7!(n-7)!}{n!} $$

$$ \therefore 3^1.2^1 = \frac{7!.(n-7)(n-8)!}{8.7!.(n-8)!} $$

$$ \therefore (6)(8) = (n-7) $$

$$ \therefore n = 55 $$

20) In the expansion of $(1+x)^{20}$, the coefficient of the r^th term is to that of the (r+1)^th term is in the ratio 1:2. Find the value of r.

Solution:

Compare given expression with $(x+a)^n$

So, $n = 20, x = 1, a = x, r = r, r^{th}\_term = T_r \Rightarrow r = r - 1$

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

$T_{r-1+1} = {}^{20}C_{r-1} (1)^{20-(r-1)} (x)^{r-1}$

$T_r = {}^{20}C_{r-1} (1)^{21-r} (x)^{r-1}$

………………………………………………….. (1)

Now, $(r+1)^{th}\_term = T_{r+1} \Rightarrow r = r$

$T_{r+1} = {}^{n}C_r x^{n-r} a^r$

$T_{r+1} = {}^{20}C_r (1)^{20-r} (x)^r$ …………………………………………………… (2)

It is given that coefficient of the r^th term is to that of the (r+1)^th term is in the ratio 1:2. So, from (1) and (2) we can say that

Coefficient of r th Term : Coefficient of (r+1) th Term =1:2

$$\frac{Coefficint\_of\_r^{th}\_Term}{Coefficint\_of\_(r+1)^{th}\_Term} = \frac{1}{2}$$

$$\therefore \frac{^{20}C_{r-1}(1)^{21-r}}{^{20}C_r(1)^{20-r}} = \frac{1}{2}$$

$$\therefore \frac{20!}{(r-1)!(20-r+1)!}.\frac{r!(20-r)!}{20!} = \frac{1}{2}$$

$$\therefore \frac{r.(r-1)!.(20-r)!}{(r-1)!.(21-r).(20-r)!} = \frac{1}{2}$$

$$\therefore \frac{r}{(21-r)} = \frac{1}{2} \Rightarrow 2r = 21-r$$

$$\therefore 3r = 21 \Rightarrow r = 7$$

21) Write down the fourth term in the binomial expansion of the function $\left(px+\frac{q}{x}\right)^n$.

i. If this term is independent of x, find the value of n.

ii. With this value of n, calculate the value of p and q given that the fourth term is equal to 160, both p and q are positive and p – q = 1.

Solution:

Compare given expression with $(x+a)^n$

So, $n = n, x = px, a = \frac{q}{x}, Fourth\_Term\_T_4 \Rightarrow r = 3$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{3+1} = {}^nC_3 (px)^{n-3} \left(\frac{q}{x}\right)^3$$

$$T_4 = {}^nC_3 p^{n-3} x^{n-3} . \frac{q^3}{x^3}$$

$$T_4 = {}^nC_3 p^{n-3} q^3 x^{n-6}$$

………………………………………………….. (1)

- If fourth term is independent of x then from equation (1) we can say that

$$x^{n-6} = x^0$$

$\therefore n-6=0$
$\therefore n=6$

- Now here n = 6 and fourth term is equals to 160. So, from equation (1) we can say that

$$T_4 = {}^6C_3 p^{6-3} q^3 x^{6-6} = 160$$
$$\therefore \frac{6!}{3!.(6-3)!} p^3 q^3 x^0 = 160$$
$$\therefore \frac{6\times5\times4\times3!}{3!(1\times2\times3)} p^3 q^3 = 160$$
$$\therefore \frac{5\times4}{1} p^3 q^3 = 160 \Rightarrow p^3 q^3 = \frac{160}{20}$$
$$\therefore p^3 q^3 = 8 \Rightarrow pq = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. (2)$$

Also, p and q are positive and p – q = 1. So, from equation (2) we can say that

$$\therefore (1+q)q = 2$$
$$\therefore q^2 + q - 2 = 0$$
$$\therefore (q+2)(q-1) = 0$$

$\therefore q=-2$ or $\therefore q=1$ but as per the given conditions p and q are positive. So, q =1 and p = 2. (∵ p – q = 1)

22) The first three terms in the expansion of $(a+b)^n$ are 1, 14 and 84 respectively. Determine a, b and n.

Solution:

Compare given expression with $(x+a)^n$

So, $n=n, x=a, a=b$

For the first term r = 0

Now, general term of the binomial expansion is

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$
$$T_{0+1} = {}^nC_0 a^{n-0} b^0 = a^n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, for the second term r = 1

$$T_{1+1} = {}^nC_1 a^{n-1} b^1 = n.a^{n-1}.b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Same as for third term r = 2

$$T_{2+1} = {}^nC_2 a^{n-2} b^2 = \frac{n.(n-1)}{1\times 2}.a^{n-2}.b^2$$ ………………………….. (3)

But it is given that $T_1 = 1, T_2 = 14, T_3 = 84$

So, from equation (1), (2) and (3) we can say that

$T_1 = a^n = 1$	$T_2 = n.a^{n-1}.b = 14$	$T_3 = \frac{n.(n-1)}{1\times 2}.a^{n-2}.b^2 = 84$
$\therefore a^n = 1$	$\therefore n.a^{n-1}.b = 14$	$\therefore \frac{n.(n-1)}{2}.(1)^{n-2}.b^2 = 84......(\because (4))$
$\therefore a = 1$… (4)	But from (4) a = 1	$\therefore n(n-1).b^2 = 168$
	$\therefore n.(1)^{n-1}.b = 14$	But from (5) b = 14 / n
	$\therefore n.b = 14$	$\therefore n(n-1).\left(\frac{14}{n}\right)^2 = 168$
	$\therefore b = 14/n$ …. (5)	$\therefore \left(\frac{n(n-1).(14)}{n^2}\right) = 12$
		$\therefore (n^2 - n)(14) = 12n^2$
		$\therefore 14n^2 - 14n - 12n^2 = 0$
		$\therefore 2n^2 - 14n = 0$
		$\therefore 2n(n-7) = 0$
		$\therefore 2n = 0$ or $\therefore n - 7 = 0$
		$\therefore n = 0$ or $\therefore n = 7$ …… (6)

Now from equation (5) b = 14/ 7 = 2 $(\because (6))$.

So, a = 1, b = 2, n = 7

23) If in the expansion of $(1+x)^n$, the fifth term be four times the fourth term and the fourth term be six times the third term. Find the value of n and x.

Solution:

For the given examples $T_5 = 4T_4$ and $T_4 = 6T_3$ ………………….. (1)

Now, Compare given expression with $(x+a)^n$

So, $n = n, x = 1, a = x$

For the third term r = 2

Now, general term of the binomial expansion is

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_{2+1} = {}^nC_2(1)^{n-2}(x)^2$$

$$\therefore T_3 = {}^nC_2x^2 \quad \text{...} \ (2)$$

For the fourth term r = 3

$$T_{3+1} = {}^nC_3(1)^{n-3}(x)^3$$

$$\therefore T_4 = {}^nC_3x^3 \quad \text{...} \ (3)$$

For the fifth term r = 4

$$T_{4+1} = {}^nC_4(1)^{n-4}(x)^4$$

$$\therefore T_5 = {}^nC_4x^4 \quad \text{...} \ (3)$$

Now, from equation (1), (2), (3) and (4)

${}^nC_4x^4 = 4\left({}^nC_3x^3\right)$	${}^nC_3x^3 = 6\left({}^nC_2x^2\right)$
$\therefore \frac{x^4}{x^3} = \frac{4.{}^nC_3}{{}^nC_4}$	$\therefore \frac{x^3}{x^2} = \frac{6.{}^nC_2}{{}^nC_3}$
$\therefore x = \frac{4.n!}{3!.(n-3)!}.\frac{4!.(n-4)!}{n!}$	$\therefore x = \frac{6.n!}{2!.(n-2)!}.\frac{3!.(n-3)!}{n!}$
$\therefore x = \frac{4.4.3!.(n-4)!}{3!.(n-3)(n-4)!}$	$\therefore x = \frac{6.3.2!.(n-3)!}{2!.(n-2)(n-3)!}$
$\therefore x = \frac{16}{(n-3)}$ .... (5)	$\therefore x = \frac{18}{(n-2)}$ .... (6)

From (5) and (6)

$$\therefore \frac{16}{(n-3)} = \frac{18}{(n-2)}$$

$$\therefore \frac{8}{(n-3)} = \frac{9}{(n-2)}$$

$$\therefore 8.(n-2) = 9.(n-3)$$

$$\therefore 8n - 16 = 9n - 27$$

$$\therefore n = 27 - 16$$

$\therefore n = 11$ and from equation (5) or (6) x=2

24) Show that the coefficient of x^5 in the expansion of $\left(1+3x\right)^4.\left(1-x\right)^3$ is 27.

Solution:

Let a = (1+3x)

So, $\left(1+3x\right)^4.\left(1-x\right)^3 = a^4.\left(1-x\right)^3$

$$\therefore a^4.\left(1-x\right)^3 = a^4\left(1-3x+3x^2-x^3\right)$$

But ,

$$a=(1+3x)$$

$$\therefore a^4.\left(1-x\right)^3 = \left(1+3x\right)^4\left(1-3x+3x^2-x^3\right)$$

$$=\left(1+4(3x)+6(3x)^2+4(3x)^3+(3x)^4\right)\left(1-3x+3x^2-x^3\right)$$

$$=\left(1+12x+54x^2+108x^3+81x^4\right)\left(1-3x+3x^2-x^3\right)$$

$$=1-3x+3x^2-x^3+2-36x+36x^2-12x^4+54x^2$$

$$-162x^3+162x^4-54x^5+108x^3-324x^4+324x^5$$

$$-108x^3+81x^4-243x^5+243x^6-81x^7$$

So, the above expansion we can find the coefficient of x^5 is $324x^5-54x^5-243x^5=27x^5$

25) Find the value of $\left(1.1\right)^5$

Solution:

$$\left(1.1\right)^5 = \left(1+0.1\right)^5$$

$$= {}^5C_0(1)^5(0.1)^0 + {}^5C_1(1)^4(0.1)^1 + {}^5C_2(1)^3(0.1)^2 +$$

$$+{}^5C_3(1)^2(0.1)^3 + {}^5C_4(1)^1(0.1)^4 + {}^5C_5(1)^0(0.1)^5$$

$$=1+5(0.1)+10(0.1)^2+10(0.1)^3+5(0.1)^4+(0.1)^5$$

$$=1+0.5+0.1+0.01+0.0005+0.00001$$

$$=1.61051$$

26) Find the value of $\left(0.81\right)^3$

Solution:

$$\left(\left(0.9\right)^2\right)^3 = \left(0.9\right)^6 = \left(1-0.1\right)^6$$

$$= {}^6C_0(1)^6(-0.1)^0 + {}^6C_1(1)^5(-0.1)^1 + {}^6C_2(1)^4(-0.1)^2 +$$
$$+{}^6C_3(1)^3(-0.1)^3 + {}^6C_4(1)^2(-0.1)^4 + {}^6C_5(1)^1(-0.1)^5 + {}^6C_6(1)^0(-0.1)^6$$
$$=1+6(-0.1)+15(-0.1)^2+20(-0.1)^3+15(-0.1)^4+6(-0.1)^5+(-0.1)^6$$

$$=1-0.6+0.15-0.02+0.0015-0.00006+0.000001$$
$$=0.531441$$

4.8 Exercise

✓ Theoretical questions

1) Explain binomial expansion in detail.
2) Give statement of binomial expansion theorem.
3) State the properties of binomial expansion theorem.
4) State the uses of binomial expansion theorem.
5) Explain how the pascal's triangle is applicable to expand the given polynomial.
6) State the properties of Pascal's triangle.
7) State the uses of Pascal's triangle.

✓ MCQs

1) The coefficient of the middle term in the expansion of $(2+3x)^4$ is:
   a) 5!
   b) 6
   c) 216
   d) 8!
   Answer: (c) 216

2) The value of $(126)^{1/3}$ up to three decimal places is
   a) 5.011
   b) 5.012
   c) 5.013
   d) 5.014
   Answer: (c) 5.013

3) If n is even in the expansion of $(a+b)^n$, the middle term is:
   a) n^{th} term
   b) $(n/2)^{th}$ term
   c) $[(n/2)-1]^{th}$ term
   d) $[(n/2)+1]^{th}$ term
   Answer: (d) $[(n/2)+1]^{th}$ term

4) The largest coefficient in the expansion of $(1+x)^{10}$ is:
   a) $10! / (5!)^2$
   b) 10! / 5!

c) 10! / (5!×4!)^2
d) 10! / (5!×4!)
Answer: (a) 10! / (5!)^2

5) The coefficient of x^3y^4 in $(2x+3y^2)^5$ is
a) 360
b) 720
c) 240
d) 1080
Answer: (b) 720

6) The largest term in the expansion of $(3+2x)^{50}$, when x = ⅕ is
a) 6th term
b) 7th term
c) 8th term
d) None of the above
Answer: (a) 6th term

7) The coefficient of y in the expansion of $(y^2+(c/y))^5$ is:
a) 10c
b) 29c
c) $10c^3$
d) $20c^3$
Answer: (c) $10c^3$

8) The fourth term in the expansion of $(x-2y)^{12}$ is:
a) $-1760\ x^9 \times y^3$
b) $-1670\ x^9 \times y^3$
c) $-7160\ x^9 \times y^3$
d) $-1607\ x^9 \times y^3$
Answer: (a) $-1760\ x^9 \times y^3$

9) If the fourth term of the binomial expansion of $(px+(1/x))^n$ is 5/2, then
a) n=6, p=6
b) n=8, p=6
c) n=8, p= ½
d) n=6, p=½
Answer: (d) n=6, p=½

10) If n is the positive integer, then $2^{3n} - 7n - 1$ is divisible by
a) 7
b) 10
c) 49
d) 81
Answer: (c) 49

✓ **Terminologies**

1) For given polynomial $(x+a)^n$ where n ϵ N can be expand as follows

$$(x+a)^n = {}^nc_0x^na^0 + {}^nc_1x^{n-1}a^1 + {}^nc_2x^{n-2}a^2 + {}^nc_3x^{n-3}a^3 + \ldots + {}^nc_nx^{n-n}a^n$$

2) The general form of the Binomial Theorem can be considered as

$$T_{r+1} = {}^{n}c_{r} x^{n-r} a^{r}$$

3) Middle terms in Binomial Expansion can be find out as follows where n is the power of the given expression:

- If the value of the n is even then middle term will be $\left(\frac{n}{2}+1\right)$
- If the value of the n is odd then middle terms will be $\left(\frac{n+1}{2}, \frac{n+1}{2}+1\right)$

✓ **Practical examples**

1) Find the value of $(31)^5$ by binomial expansion theorem.

Answer: 2,86,29,151

2) Find the value of $(99)^4$ by binomial expansion theorem.

Answer: 9,60,59,601

3) Find the value of $(19.9)^5$ by binomial expansion theorem.

Answer: 31,20,796.00999

4) Find the value of $\left(\sqrt{3}+2\right)^6+\left(\sqrt{3}-2\right)^6$ with binomial expansion theorem.

Answer:2702

5) Find the value of $\left(\sqrt{7}+1\right)^5+\left(\sqrt{7}-1\right)^5$ with binomial expansion theorem.

Answer: 632

6) Obtain the fourth term in the expansion of $\left(\frac{x}{2y}+\frac{2y}{x}\right)^{10}$

Answer: $\frac{15x^4}{2y^4}$

7) Obtain the eighth term in the expansion of $\left(\frac{3x}{2}-\frac{2}{3x}\right)^{11}$

Answer: $-\frac{880}{9x^3}$

8) Obtain the middle term in the expansion of $\left(\frac{x^2}{2}-\frac{2}{x}\right)^{8}$

Answer: Middle term $T_5 = 70x^4$

9) Obtain the middle terms in the expansion of $\left(\frac{x}{3}-\frac{3}{x}\right)^{7}$

Answer: Middle term $T_4 = -\frac{35x}{3}$ and $T_5 = \frac{105}{x}$

10) Obtain the term with x^9 in the expansion of $\left(2x^2 + \frac{1}{2x}\right)^9$

Answer: $T_4 = 672x^9$

11) Obtain the constant term in the expansion of $\left(x - \frac{2}{x^2}\right)^{12}$

Answer: $T_5 = 7920$

12) Obtain coefficient of x in the expansion of $\left(x - \frac{2}{x}\right)^9$

Answer: 2016

PART - 2

BBA

SEMESTER-1

BUSINESS MATHEMATICS

BLOCK: 2

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Publisher's Name: Dr. Ajaysinh Jadeja,
Registrar,
Dr. Babasaheb Ambedkar Open University
'JyotirmayParisar',Opp.ShriBalaji
Temple,Chharodi,Ahmedabad,382481,
Gujarat,India.
Edition: 2024 (First Edition)
ISBN: **978-93-5598-619-1**

978-93-5598-619-1

UNIT-5 ARITHMETIC AND GEOMETRIC PROGRESSION

5.1 Overview:

There is a list of activities which have their own sequence in the series i.e., collection of objects and their order in such a way that it has known first member, second member and so on. Some examples of sequence are population of human beings over a period of time, bacteria at different times, amount of money deposited in a bank over a number of years and Depreciated values of fixed assets. In simple words we can say that Sequences which follow specific patterns are called progressions and particular types/sets of it known as arithmetic progressions (APs), Geometric progressions (GPs) and Harmonic progression (HPs).

5.2 Meaning:

5.2.1 Meaning of Sequences:

It is a set of numbers which are written in some particular order. in simple words we can say that sequence is a form / order of number or objects. Take an illustration of given number. 2, 4, 6, 8, 10, ...

Here, we look to have a fixed order of numbers, which is a sequence of even numbers. In other words, we start with the number 2, which is an even number, and then each successive number is obtained through adding 2 to give the next even number.

2, 4, 6, 8, 10, ...   is a sequence of numbers. The dots written at the end specify that we consider this sequence as an infinite sequence because it goes on forever. On the other side we can take only five number of the above mention sequence or series of number i.e., 2, 4, 6, 8, 10. This is called a finite sequence containing just five numbers. 2, 4, 6, 8, 10, . . .n. These numbers can use for counting and have included n at the end of sequence. The last number n after the dots tells us about finite sequence.

The First term in a sequence called u_1, the second term u_2, the third term u_3, and at the end of the sequence or n^{th} term, write as a u_n. Now, the sequence as per notation would be $u_1, u_2, u_3, . . . , u_n$.

5.2.2 Meaning of Series:

A series is called the addition of all the sequence. In simple words, we can say that series is obtained from a sequence by adding all the terms/numbers of together.

Take a look on illustration that we have the sequence $u_1, u_2, u_3, . . . , u_n$. The series obtained from this sequence is the addition of $u_1 + u_2 + u_3 + . . . + u_n$, and S_n is the sum of the n^{th} terms. So, we can correlate the ideas of sequence and series. there is an important distinction between them.

let us take one illustration consider the sequence of numbers 1, 2, 3, 4, 5, 6, . . . , n.

Then $S_1 = 1$ is the sum of the first term,

The $S_2 = 1 + 2 = 3$ sum of the first two terms,

$S_3 = 1 + 2 + 3 = 6$,

$S_4 = 1 + 2 + 3 + 4 = 10$,

$S_5 = 1 + 2 + 3 + 4 + 5 = 15$ and so on.

If there are n terms in the sequence then the sum of n^{th} number (Sn) for the result,

$$\mathbf{Sn = u_1 + u_2 + u_3 + . . . + u_n.}$$

5.3 Types of progressions:

- Arithmetic Progression,
- Geometric Progression and
- Harmonic progression

1. Arithmetic Progression: it is a sequence where each new term/number after the first is getting by adding a constant common difference to the preceding term called d. If the first term of the sequence is a and common difference is d then the arithmetic progression is

   $a, a + d, a + 2d, a + 3d, a + 4d, ... a + (n - 1)d.$

2. Geometric Progression: it is a sequence where each new term/ number after the first is getting by multiplying the preceding term by the constant common ratio called r. If the first term of the sequence is known as a then the geometric progression is

a, ar, ar2, ar3, ar4, ar5 . . . ar^{n-1}.

3. Harmonic progression: it is a sequence of numbers in which the reciprocals of the term of an AP. First term of HP is a and common difference is d. the general formulation of HP is as under.

1/a, 1/ (a + d), 1/ (a + 2d), ………

5.4 Meaning and Definitions of Arithmetic progression (AP) :

let's us understand with the series 1, 3, 5, 7, . . .

This sequence starts with a particular first term, and get successive terms by add a fixed value to the previous term. In the first number /term add 2 to get the next term. So, the difference of the two consecutive terms in each sequence is a constant. Any sequence with this property is called an arithmetic progression (AP). The successive terms are obtained through adding a constant number to the preceding terms. The list of such terms or numbers is formed an Arithmetic Progression (AP). The constant number is called the common difference of the AP. it can be positive, negative or zero.

Now, use algebraic notation to represent an arithmetic progression (AP). The first term of the sequence is denoted by a and the common difference between successive terms or numbers denoted by d. The first term is denoted by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes a_1, a_2, a_3, . . ., a_n.

The value of d can be found through = $a_3 - a_2 = a_2 - a_1 = a_n - a_{n-1} = d$.

let's us understand with the series 1, 3, 5, 7, . . . and represent in the algebraic form.

1, 1+(1 x 2), 1+(2 x 2), 1+(3 x 2), …………

a, a + d, a + 2d, a + 3d, .

where, the first term a = 1, and the common difference d = 2. If the sequence end with n^{th} term, then the = a + (n − 1) d.

if there are n terms/numbers in the sequence then (n − 1) is the common differences between the two successive terms and add on (n − 1) d to the first term a. Last term of a finite sequence ℓ, the equation would be ℓ = a + (n − 1) d .

Illustration: - 1 From the given below AP, write the first term a and the common difference d.

1. 4, 10, 16, 22, .
2. 6, 9, 12, 15, . . . ,

Answers-: Here, first term a = 4

common difference d= $a_2 - a_1 = 10 - 4 = 6$

Illustration: - 2 From the given below AP, write the first term a and the common difference d.

1. 6, 9, 12, 15, . . . ,

Answers-: Here, first term a = 6

common difference d= $a_2 - a_1 = 9 - 6 = 3$

note: we can find d using any two consecutive terms.

Illustration: - 3 Write down first four terms from the given AP, when the first term a and the common difference d provided.

(i) a = 10, d = 10
(ii) a = 2, d = 4
(iii) a = 2, d = – 3
(iv) a = – 1, d = 4

Answers-:

(i) a = 10, d = 10

first four terms of any AP = a, a + d, a + 2d, a + 3d, .
Now, put the value of a and d in this equation = 10, 10+10, 10+10x2, 10+10x3.
= 10,20,30,40

(ii) a = 2, d = 4

first four terms of any AP = a, a + d, a + 2d, a + 3d, .
Now, put the value of a and d in this equation = 2, 2+4, 2+4x2, 2+4x3.
= 2,6,10,14

(iii) a = 2, d = – 3
first four terms of any AP = a, a + d, a + 2d, a + 3d, .
Now, put the value of a and d in this equation = 2, 2+(-3), 2+(-3) x 2, 2+(-3) x 3.
= 2, -1, -4, -7

(iv) a = – 1, d = 4
first four terms of any AP = a, a + d, a + 2d, a + 3d, .
Now, put the value of a and d in this equation = -1, -1+4, -1+4 x 2, -1+4 x 3.
= -1, 3, 7, 11

5.4.1 n^{th} term and the Sum of n^{th} term of an Arithmetic progression (AP):

It is a sum or addition of all the terms from first term to last term of the arithmetic series. The sum of the first n terms of an AP, when the nth term of this AP is a + (n – 1) d. Then will get the sum of the first n terms, which is denoted by Sn.

$$Sn = a + (a + d) + (a + 2d) + . . . + (\ell - 2d) + (\ell - d) + \ell . \qquad (1)$$
equation.

Let us write the series down again and write it down with the terms in reverse order.

$Sn = \ell + (\ell - d) + (\ell - 2d) + . . . + (a + 2d) + (a + d) + a$. (2) equation

By adding (1) and (2) equation both the side, so that each term in the first series will be added to the term vertically in the second series, we get

$2Sn = (a + \ell) + (a + \ell) + (a + \ell) + . . . + (a + \ell) + (a + \ell) + (a + \ell)$,

We can write above equation in a simplified format and in the right-hand side n number of the term $(a + \ell)$, we get

$2Sn = n(a + \ell)$.

Now, Sn rather than 2Sn, and so we divide by 2 to get

$Sn = ½ n (a + \ell)$.

when we put the value of $\ell = a + (n - 1)d$, in the equation then we get

$Sn = ½ n (a + a + (n - 1)d)$

$Sn = ½ n (2a + (n - 1)d)$.

5.4.2 Properties / Features of an Arithmetic Progression (AP):

- If a constant is added in each term of arithmetic **Progression**, then the resultant sequence is known as A.P.
- If a constant is subtracted in each term of arithmetic **Progression**, then the resultant sequence is known as A.P.
- If a constant is multiplied in each term of arithmetic **Progression**, then the resultant sequence is known as A.P.
- If a non-zero constant is divided in each term of arithmetic **Progression**, then the resultant sequence is known as A.P.
- Each term is obtained by adding a fixed constant (d) to the previous term is known as Common Difference.
- The first term is denoted by 'a'.
- The nth term is denoted by $a_n = a + (n-1) d$.
- The sum of n terms is denoted by $Sn = n/2 [2a + (n-1) d]$

Illustration: - 4 Find out the sum of the first 30 terms of the given below sequence of the arithmetic progression.

1, 3, 5, 7, 9, ...

Answer:

a = 1 , d = 2 , n = 30.

We now use the formula of the sum of nth terms of AP,

$S_n = \frac{1}{2} n (2a + (n - 1)d)$.

$S_{30} = \frac{1}{2} \times 30 (2 \times 1 + (30 - 1) 2)$.

$= 15 \times (2 + 29 \times 2)$.

$= 15 \times (2 + 58)$.

$= 900$.

Illustration: - 5 The income of a person for the first year is Rs. 2,00,000, and he receives an increase of Rs.10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Answers: A.P. with a = 2,00,000, d = 10,000, and n = 20.

We now use the formula of the sum of nth terms of AP,

$S_n = \frac{1}{2} n (2a + (n - 1) d)$.

$S_{20} = \frac{1}{2} \times 20 (2 \times 2{,}00{,}000 + (20 - 1) 10{,}000)$.

$= 10 \times (4{,}00{,}000 + 19 \times 10{,}000)$.

$= 10 \times (4{,}00{,}000 + 1{,}90{,}000)$.

$= 10 \times 5{,}90{,}000$

$= 59{,}00{,}000$

Illustration: - 6 An arithmetic progression first term is 3 and the sum of the first 8 terms is twice the sum of the first 5 terms. Find the value of d.

Answers: where, a = 3, the sums of the S8 are the twice of the sum of the S5, and common difference d = ?.

We now use the formula of the sum of nth terms of AP,

$S_n = \frac{1}{2} n (2a + (n - 1) d)$.

for the sum of the first 8 terms $(S_8) = \frac{1}{2} \times 8 (2 \times 3 + (8 - 1) d)$ and

for the sum of the first 5 terms $(S_5) = \frac{1}{2} \times 5 (2 \times 3 + (5 - 1) d)$

So, using the given fact that $S_8 = 2S_5$,

$\frac{1}{2} \times 8 (2 \times 3 + (8 - 1) d) = 2 (\frac{1}{2} \times 5 (2 \times 3 + (5 - 1) d))$

$4 (6 + 7 d) = 2 (2.5 (6 + 4 d))$

$4 (6 + 7 d) = 5 (6 + 4 d)$

$24 + 28d = 30 + 20d$

$8d = 6$

$d = 6 / 8 = 3 / 4$

Illustration: - 7 Find out the sum of first 22 terms of an AP in which d = 7 and 22nd term is 149.

Answers: in this illustration common difference d = 7, last term (22nd term) = 149 and sum of first 22 terms = ?

Formula of last term = (22nd term) = $\ell = a + (n - 1) d = 149$

when we put the value of d = 7 and n = 22 in the equation then we get

$= 149 = a + (n - 1) d.$

$= 149 = a + (21) 7.$

$= 149 = a + 147.$

Value of a = 2, put the value in the equation

$Sn = ½ n (a + \ell).$

$S22 = ½ x 22 (2 + 149) .$

$= 11 (151) = 1882$

Illustration- 8 The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, find out how many terms are there and what is their sum?

Answers: in this illustration, first term (a) = 17 common difference d = 9, last term = 350 and sum of first nth terms = ?

Formula of last term = $\ell = a + (n - 1) d = 350$

when we put the value of d = 7 and a = 17 in the equation then we get

$\ell = a + (n - 1) d = 350$

$= 350 = a + (n - 1) d.$

$= 350 = 17 + (n - 1) 9.$

$= 350 - 17 = (n - 1) 9.$

$= 333 = 9n - 9.$

$= 333 + 9 = 9n$

$= 342 = 9n$

$n = 38$

Value of a = 17 and n = 38 put the value in the equation sum of 38th terms = S38 = ½ n (a + ℓ).

S38 = ½ x 38 (17 + 350) .

= 19 (367) = 6973

5.5 Meaning and Definitions of Geometric progressions (GP):

It is the other type of sequence in which each term is obtained by multiplying a fixed number or constant in the previous term.

Let us understand with the sequence

2, 6, 18, 54, ...

In the above mention sequence each term in the sequence is the multiplication of fixed or constant term to the previous term. These types of Sequences are known as geometric progressions, or GPs.

A sequence a_1, a_2, a_3, a_4 …, a_n, … is called geometric progression, if each term is non-zero and a_1 = a, is first term of sequence which is called a in a geometric progression. In AP, there is common difference d Instead of the common ratio in GP as the ratio of successive terms which is always constant and known as $r = a_{K+1} / a_K$. The general geometric progression with notation is as follows:

a, ar, ar2, ar3, … ., where

a = first term of GP and

r = common ratio of the G.P.

The nth term can be calculated by ar^{n-1}, where the power (n − 1) is always one less than the position n of the term in the sequence. Let's understand through the example

2, 6, 18, 54, ...

In the above sequence a = 2 and r = second term / first term = 6 / 2 = 3,

we could write the sequence in the GP as

$2, 2 \times 3, 2 \times 3^2, 2 \times 3^3, 2 \times 3^4. . . .$

Note: the first term is a and second term a_2 is obtained by multiplying first term and common ratio = a x r = ar, same way we can obtained more term like:

2nd term = a_2 = first term x common ratio = ar = ar^{2-1}

3rd term = a_3 = second term x common ratio = a_2 x r = $ar^2 = ar^{3-1}$

4th term = $a_4 = ar^3 = ar^{4-1}$, and so on till nth term.

5.5.1 n^{th} term and the Sum of n^{th} term of a Geometric progression (GP):

It is a sum of all the terms of geometric progression. In simple word it is the sum of the first n terms of a geometric progression which is denoted by Sn.

Sn = a + ar + ar2 + ar3 + . . . + arn−1. Equation (1)

Each consecutive or next term is getting by the multiplication of r common ratio with previous term.

Multiple both the side with r common ratio, then we got the following equation:

rSn = ar + ar2 + ar3 + . . . + ar^n−1 + ar^n . Equation (2)

Now, subtract equation (1) from (2) then we got the following equation.

$$Sn - rSn = a \quad . . . \quad - ar^n .$$

$$Sn (1 - r) = a (1 - r^n) .$$

$$Sn = a (1 - r^n) / (1 - r)$$

5.5.2 Features / Properties of the Geometric Progression (GP):

- Each term is obtained by multiplying a fixed constant (r) to the previous term is known as Common Ratio.
- The first term is denoted by a of the GP.
- The nth term is the last term of the GP known by = ar^{n-1}
- Sum of n Terms of the GP is the sum of n terms is denoted by Sn = a (r^n - 1) / (r - 1)

Illustration: - 9 Find out the sum of the geometric series.

2 + 6 + 18 + 54 + . . .

where the value of n is 5.

Answers: in this series, the value of a = 2, r = second term / first term = 6/2 = 3 and n = 5.

the sum of the first n terms of a geometric progression which is denoted by

$$Sn = \frac{a (1 - r^n)}{(1 - r)}$$

$$Sn = \frac{2 (1 - 3^5)}{(1 - 3)}$$

$$Sn = 2 (1 - 243) / -2$$

$$= 2 (- 242) / - 2$$

$= -(-242)$

$= 242.$

Illustration: - 10 Find the sum of the geometric series of the given below.

$8 - 4 + 2 - 1 + . . .$

where there are 6 terms in the series.

Answers: in this series, the value of a = 8, r = second term / first term = - 4 / 8 = −1/2 and n = 6.

the sum of the first n terms of a geometric progression which is denoted by

$$Sn = \frac{a(1 - r^n)}{(1 - r)}$$

$$Sn = \frac{8(1 - (-1/2)^6)}{(1 - (-1/2))}$$

$$= \frac{8(1 - (-1/64)}{3/2}$$

$$= \frac{2 \times 8 \times 65}{3 \times 64}$$

$= 65 / 12 \quad = 5$ and $5/12$

Illustration: - 11 Find out the terms of the geometric progression.

2, 4, 8, . . . , 64 ?

Answers: in this series, the value of a = 2, r = second term / first term = 4 / 2 = 2 and n = ?.

the sum of the first n terms of a geometric progression which is denoted by

$$Sn = \frac{a(1 - r^n)}{(1 - r)}$$

the formula for the nth term = ar^{n-1}. Then putting the value in the formula.

$64 = ar^{n-1}$

$64 = 2 \times 2^{n-1}$

$32 = 2^{n-1}$

$2^5 = 2^{n-1}$

$5 = n - 1$

n = 6, So there are 6 terms in this geometric progression.

Illustration: - 12 Find out the first five terms of the GP with first term a = 3 and common ratio r = 2.

Answers: in this illustration the value of a = 3 and common ratio r = 2 given.

First five terms of GP are = a, ar, ar^2, ar^3, ar^4.

= 3, 3 x 2, 3 x $(2)^2$, 3 x $(2)^3$, 3 x $(2)^4$,

= 3, 6, 12, 24, 48

Illustration: - 13 Find out the sum of the first 20 terms of the GP with first term 3 and common ratio 1.5.

Answers: in this illustration the value of a = 3, common ratio r = 1.5 and value of n is 20 givens.

The formula of the sum of the first n terms of a geometric progression which is denoted by

$$Sn = \frac{a(1 - r^n)}{(1 - r)}$$

$$S_{20} = \frac{3(1 - (1.5)^{20})}{(1 - 1.5)}$$

S_{20} = 3 (1- 3325.26) / (- 0.5)

= 3 (- 3324.26) / (- 0.5)

= - 9975.77/ - 0.5

= 19,951.54

Illustration: - 14 In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 8th term of GP.

Answers: in this illustration, a_3 = 24 and a_6 = 192 is given.

a3 is equal to ar^2, which is 24 and a_6 is equal to ar^5 which is 192

Dividing a6 by a3 = a6 / a3

= 192 / 24

we get $r^3 = 8$. So, r = 2

putting the value of r in $ar^2 = 24$,

$= a (2)^2 = 24$

we get a = 6.

Hence $a_8 = ar^7 = 6 (2)^7 = 768$.

5.6 Relationship Between A.M. and G.M.:

There is a relationship between the Arithmetic Progression (AP) and Geometric Progression (GP) and we take a look on the relationship between AP and GP with the help of the examples. If a and b are two given positive real numbers and then A is the AM and G is the GM, then

$A = (a + b) / 2$ and $G = \sqrt{ab}$

Thus, we have $A – G = a + b / 2 - \sqrt{ab}$

$= a + b -2\sqrt{ab} / 2$

$= (\sqrt{a} - \sqrt{b})^2 \ / 2 \ \geq 0$

Hence, the relationship $A \geq G$.

Illustration: - 15 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors of the preceding sixth generations.

Answers: in this illustration value of a is 2, second term / first term r = 2 and n = 6

Using the sum formula

$$Sn = \frac{a (1 - r^n)}{(1 - r)}$$

$$S6 = \frac{2 (1 – (2)^6)}{(1 – 2)}$$

$= 2 (1- 64) / - 1$

$= - 126 / -1 = 126$

❖ **Exercises**

- Multiple choice questions (MCQs) of arithmetic and geometric progressions given below

1. **Common difference (d) of an Arithmetic Progression (AP) is known as:**
   a) fixed constant multiplied by each term
   b) fixed constant added to each term

c) difference between the first and last term
d) none of the above

Answer: b) fixed constant added to each term

2. **Which sequence is an example of an Arithmetic Progression (AP)?**

   a) 2, 4, 8, 16, 32, ...

   b) 2, 5, 8, 11, 14, ...

   c) 1, 3, 9, 27, ...

   d) none of the above

   Answer: b) 2, 5, 8, 11, 14, ...

3. **The common ratio (r) of a geometric progression is known as:**

   a) fixed constant multiplied by each term

   b) fixed constant added to each term

   c) difference between the first and last term

   d) none of the above

   Answer: a) fixed constant multiplied by each term

4. **Which of the following sequence is an example of a geometric progression?**

   a) 2, 4, 6, 8, ...

   b) 1, 2, 4, 8, 16, ...

   c) 1, 3, 5, 7, 9, ...

   d) none of the above

   Answer: b) 1, 2, 4, 8, 16, ...

5. **The sum of the first nth terms of an Arithmetic Progression (AP) is:**

   a) n/2 (a + l)

   b) n/2 (2a + (n-1) d)

   c) n (a + l)/2

   d) none of the above

   Answer: b) n/2 (2a + (n-1) d)

6. **The formula of nth term of an Arithmetic Progression (AP) is:**

   a) a - (n-1) d

b) a - nd

c) a + (n-1) d

d) none of the above

Answer: c) a + (n-1) d

7. **the first term (a) of an Arithmetic Progression (AP) 2 and the common difference (d) 3, find out the 6th term?**

a) 7

b) 15

c) 12

d) 14

Answer: d) 14

8. **The sum of the first nth terms of a Geometric Progression (GP) is:**

a) a $(r^n + 1) / (r + 1)$

b) a $(1 - r^n) / (1 - r)$

c) a $(r^n - 1) / (r - 1)$

d) none of the above

Answer: c) a $(r^n - 1) / (r - 1)$

9. **The nth term of a Geometric Progression (GP) is known as:**

a) ar^n

b) $ar^{(n-1)}$

c) $a/r^{(n-1)}$

d) none of the above

Answer: b) $ar^{(n-1)}$

10. **If the first term (a) of a geometric progression (GP) 4 and the common ratio (r) 2, find out the 4th term?**

a) 64

b) 32

c) 16

d) none of the above

Answer: c) 32

- **Short and long questions:**

1. Write down S1, S2, . . . , Sn for the sequences

   (a) 1, 3, 5, 7, 9, 11;

   (b) 4, 2, 0, −2, −4.

2. Write down the first five terms of the AP with first term 8 and common difference 7.
3. Write down the first five terms of the AP with first term 2 and common difference −5.
4. What is the common difference of the AP 11, −1, −13, −25, . . . ?
5. Find the 17th term of the arithmetic progression with first term 5 and common difference 2.
6. Write down the 10th and 19th terms of the APs

   (i) 8, 11, 14, . . .,

   (ii) 8, 5, 2 . . ..

7. An AP is given by k, 2k/3, k/3, 0, ...

   (i) Find the sixth term.

   (ii) Find the n th term.

   (iii) If the 20th term is equal to 15, find k.

8. Find the sum of all natural numbers lying between 100 and 1000, which are multiples of 5.
9. In an A.P., the first term is 2 and the sum of the first five terms is one-fourth of the next five terms. Show that 20th term is –112.
10. Find the 9th and nth terms of the G.P. 5, 25, 125, …. .
11. 11.The 4th term of a G.P. is square of its second term, and the first term is – 3. Find out the 7th term.
12. Given a G.P. with a = 729 and 7th term 64, find out the S7.
13. Find out the sequence if a sum of the first two terms of a G.P. is – 4 and the fifth term is 4 times the third term.
14. What will Rs 5000 amounts to in 5 years after its deposit in a bank which pays interest rate of 10% annually?
15. Find out the sum of first 21 terms of an AP in which d = 7 and 22nd term is 147.
16. Find out the value of n and Sn in the AP when the value of a = 5, d = 3, an = 50 is given.
17. Find out the value of d and S13 in the AP when the value of a = 7, a13 = 35 is given.
18. Find out the value of a and S12 in the AP when the value of a12 = 37, d = 3 is given.

19. Find out the value of d and a10 in the GP when the value of a3 = 15 and S10 = 125 is given.
20. find the value of a and a9 in the GP when the value of d = 5, S9 = 75 is given.

UNIT-6	DETERMINANTS

6.1 Definition

Determinant is a system of representing numbers in a square arrangement under a particular symbol (between vertical bars).

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

6.2 Value of the 2 x 2 and 3 x 3 determinants

➢ **Value of determinants of order 2:**

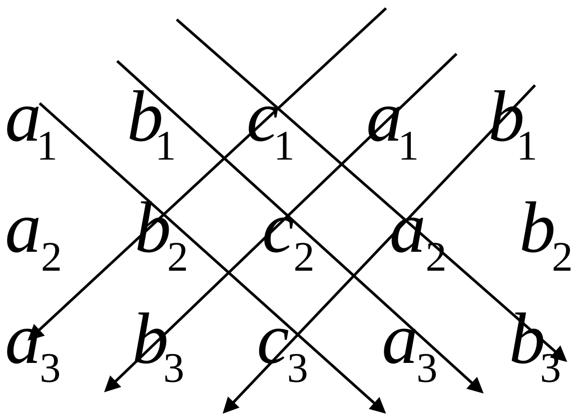
Suppose D is the given 2 X 2 determinants then its value can be found out as follows:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

➢ **Value of determinants of order 3:**

1) The Sarrus Method:

Suppose $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the method can be applied as follows to get determinant value of 3 x 3 problem



The product of each element joined by downwards arrows followed by plus sign whereas product of each element joined by upwards arrows followed by minus sign. So, all together it can be considered as follows

$|D| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$

2) By method of minors and co – factors:

- **Definitions:**

i. **Minor:** The Minor of an element in a determinant is a determinant that is left after removing the row and the column which intersect at the element and is of order one less than that of the given determinant.

ii. **Co – factor:** If we apply the appropriate sign to the minor of an element then we have its co – factor.

Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$ Where M_{ij} is Minor of a_{ij}

Determinant of $D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$|D| = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= 1(45-48) - 2(36-42) + 3(32-35)$

$= -3 + 12 - 9$

$= 0$

6.3 Properties of determinant

1) If each entry in any row or each entry in any column of a determinant is 0 (zero) then the determinants is equal to 0 (zero).

$|D| = \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} = 0$

2) If rows be changed into columns and columns into rows, the determinants remains unaltered.

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$

3) If any two rows (or columns) of a determinant are interchanged, the resulting determinant is the negative of the original determinant.

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$

4) If two rows (or two columns) in a determinant are identical, the determinant is equal to zero.

$|D| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

5) If all the elements of any row (or column) be multiplied by a non zero real number k, then the value of the new determinant is 'k' times the value of the original determinant.

$|D| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$ Suppose each entry of first row is multiplied by 5 then

$$|D| = \begin{vmatrix} 5 & 10 \\ 3 & 4 \end{vmatrix} = 5(-2) = -10$$

6) If each entry in a row (or column) of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants.

$$\begin{vmatrix} 1+2 & 3 \\ 4+5 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

7) If each entry of one row (or column) of a determinant is multiplied by a real number 'k' and the resulting product is added to the corresponding entry in another row (or column respectively) in the determinant, then the resulting determinant is equal to the original determinant.

$|D| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$ Suppose we perform R1 + 2R2 then also

$$|D| = \begin{vmatrix} 1+6 & 2+8 \\ 3 & 4 \end{vmatrix} = -2$$

6.4 Cramer's rule (two and three variables)

- **Determinant solution of linear equations (Cramer's rule) for two variables:**

1) Consider the simultaneous equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Solving these equations, we get

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

OR

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}$$

Where $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ also $D \neq 0$

- **Determinant solution of linear equations (Cramer's rule) for three variables:**

2) Consider the simultaneous equations

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Solving these equations, we get

$$x=\frac{\begin{vmatrix}d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix}}{\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\ y=\frac{\begin{vmatrix}a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix}}{\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}},\ z=\frac{\begin{vmatrix}a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix}}{\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}$$

OR

$$x=\frac{D_x}{D},\ y=\frac{D_y}{D},\ z=\frac{D_z}{D}$$

Where,

$$D_x=\begin{vmatrix}d_1 & b_1 & c_1\\ d_2 & b_2 & c_2\\ d_3 & b_3 & c_3\end{vmatrix},D_y=\begin{vmatrix}a_1 & d_1 & c_1\\ a_2 & d_2 & c_2\\ a_3 & d_3 & c_3\end{vmatrix},D_z=\begin{vmatrix}a_1 & b_1 & d_1\\ a_2 & b_2 & d_2\\ a_3 & b_3 & d_3\end{vmatrix},D=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$$

Also $D\neq 0$

6.5 Illustrations

1) Compute determinant of $D=\begin{vmatrix}-6 & 7\\ -9 & 8\end{vmatrix}$

Solution:

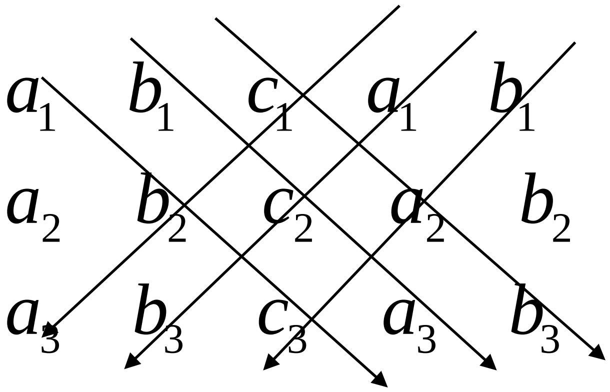
$$|D|=(-6)(8)-(7)(-9)=-48+63=15$$

2) Compute determinant value of $D=\begin{vmatrix}3 & 2 & 1\\ 4 & 1 & -7\\ 0 & 3 & 4\end{vmatrix}$ by Sarrus Method.

Solution:

The Sarrus Method can be used for the given 3 x 3 determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ as follows}$$



The product of each element joined by downwards arrows followed by plus sign whereas product of each element joined by upwards arrows followed by minus sign. So, all together it can be considered as follow

$$|D| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

= 55 (by comparing with the given determinant)

3) Compute the determinant of $D = \begin{vmatrix} 7 & 2 & 3 \\ 6 & 4 & 5 \\ 1 & 8 & 9 \end{vmatrix}$

Solution:

$$|D| = 7\begin{vmatrix} 4 & 5 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 6 & 5 \\ 1 & 9 \end{vmatrix} + 3\begin{vmatrix} 6 & 4 \\ 1 & 8 \end{vmatrix}$$

$$= 7(36-40) - 2(54-5) + 3(48-4)$$

$$= -28 - 98 + 132$$

$$= 6$$

4) Solve following simultaneous equations using Cramer's Method.

2x – y – 5 = 0

3x + 2y + 3 = 0

Solution:

Consider the simultaneous equations in the form of

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

So,

2x – y = 5

3x + 2y = - 3

Then solve these equations as follows

$$x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},\ y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

OR

$$x=\frac{D_x}{D}, y=\frac{D_y}{D}$$

Where $D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ also $D\neq 0$

Let us first find out the denominator of the quotient of the value x and y as follows:

$$D=\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}=(2).(2)-(-1).(3)=4+3=7$$

So, $D\neq 0$ and the system has a unique solution.

$$D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=\begin{vmatrix} 5 & -1 \\ -3 & 2 \end{vmatrix}=(5).(2)-(-1).(-3)=10-3=7$$

$$D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=\begin{vmatrix} 2 & 5 \\ 3 & -3 \end{vmatrix}=(2).(-3)-(5).(3)=-6-15=-21$$

Now,

$$x=\frac{D_x}{D}=\frac{7}{7}=1, y=\frac{D_y}{D}=\frac{-21}{7}=-3$$

5) Solve following simultaneous equations using Cramer's Method.

2x + y – z = 3

x + y + z = 1

x – 2y – 3z = 4

Solution:

Our simultaneous equations are in the form of

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Solving these equations, we get

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

OR

$$x=\frac{D_x}{D}, y=\frac{D_y}{D}, z=\frac{D_z}{D}$$

Where,

$$D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Also $D \neq 0$

Let us first find out the denominator of the quotient of the value x, y and z as follows:

$$D=\begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix}=2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}-1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}-1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$=2\left[(1).(-3)-(1).(-2)\right]-1\left[(1).(-3)-(1).(1)\right]-1\left[(1).(-2)-(1).(1)\right]$$

$$=2(-3+2)-1(-3-1)-1(-2-1)$$

$$=-2+4+3$$

$$=5$$

So, $D \neq 0$ and the system has a unique solution.

$$D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix}=10$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}=\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix}=-5$$

$$D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}=\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}=0$$

So, the required solution is

$$x=\frac{D_x}{D}=\frac{10}{5}=2$$

$$y=\frac{D_y}{D}=\frac{-5}{5}=-1$$

$$z=\frac{D_z}{D}=\frac{0}{5}=0$$

6) Prove that $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=(a-b).(b-c).(c-a)$

Solution:

Apply C1 – C2, C2 – C3

$$L.H.S.=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}=\begin{vmatrix}0 & 0 & 1\\ a-b & b-c & c\\ a^2-b^2 & b^2-c^2 & c^2\end{vmatrix}$$

$$=(a-b).(b-c).\begin{vmatrix}0 & 0 & 1\\ 1 & 1 & c\\ a+b & b+c & c^2\end{vmatrix}$$

Now, expanding by first row

$$=(a-b).(b-c).1\begin{vmatrix}1 & 1\\ a+b & b+c\end{vmatrix}$$

$$=(a-b).(b-c).(c-a)$$

$$=R.H.S.$$

6.6 Exercise

✓ Theoretical questions

1) Define determinant with proper example.
2) Explain the properties of determinants in details.
3) Explain Sarrus method for getting determinant value for 3 X 3 problem.
4) Explain Cramer’s rule for two variables linear equations.
5) Explain Cramer’s rule for three variables linear equations.

✓ MCQs

1) If $A=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}, B=\begin{vmatrix}1 & bc & a\\ 1 & ca & b\\ 1 & ab & c\end{vmatrix}$ then

a) Det. A + Det. B = 0

b) Det. A + 2(Det. B) = 0

c) Det. A = Det. B

d) None of these

Answer: a

2) The value of $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is

a) 5^2

b) 0

c) 5^{13}

d) 5^9

Answer: b

3) The system of equation x + 2y = 11, - 2x – 4y = 22 has:

a) Only one solution

b) Finitely many solutions

c) No solution

d) Infinitely many solutions

Answer: c

4) Value of $\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}$ is

a) 2

b) 6

c) 24

d) 120

Answer: 24

5) Value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ is

a) x + y

b) x - y

c) xy

d) none of these

Answer: c

6) Value of $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$ is

a) 0
b) 1
c) - 1
d) None of these

Answer: a

7) The Sarrus method applicable for

a) 2 x 2 determinants
b) 3 x 3 determinants
c) Any order determinants
d) None of these

Answer: b

8) Through Cramer's method simultaneous linear equations can be solved

a) Yes
b) No
c) Can't say
d) None of these

Answer: a

9) Value of $\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$ is

a) – 2
b) 2
c) $x^2 - 2$
d) $x^2 + 2$

Answer: a

10) Value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is

a) a b c (a + b + c)
b) $a^3 + b^3 + c^3$ -3 a b c
c) - a^3 - b^3 - c^3 + 3 a b c
d) None of these

Answer: c

✓ **Practical examples**

1) Without expanding find the value of $\begin{vmatrix} 2 & 45 & 55 \\ 1 & 92 & 32 \\ 3 & 68 & 87 \end{vmatrix}$

Answer: 54

2) Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

3) Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (b-c).(c-a).(a-b)$$

4) Prove that

$$\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$$

5) Prove that

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

6) Prove that

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

7) Prove that

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

8) Prove that

$$\begin{vmatrix} x-y & 1 & x \\ y-z & 1 & y \\ z-x & 1 & z \end{vmatrix} = \begin{vmatrix} x & 1 & y \\ y & 1 & z \\ z & 1 & x \end{vmatrix}$$

9) Solve following

$2x + 5y = 9$

$3x - 2y = 4$

Solution:

$x = 2, y = 1$

10) Solve following

$4x - y = 2$

$x - 3y = -5$

Solution:

$x = 1, y = 2$

11) Solve following

$2x - 3y + z = 3$

$x + y - 2z = -1$

$3x - 2y + 2z = 8$

Solution:

$x = 2, y = 1, z = 2$

12) Solve following

$2x + y - z = 3$

$x + y + z = 1$

$x - 2y - 3z = 4$

Solution:

$x = 2, y = -1, z = 0$

UNIT-7 MATRIX

7.1 Definition:

Matrix:

A matrix (plural is matrices) is an array of real numbers (or other suitable entities), arranged in rows and columns.

7.2 Terminology associated to matrix

1) The order of the matrix:

$A = \left[a_{ij} \right]_{mn}$ where i = 1, 2, 3, … ,m; j = 1, 2, 3, … ,n.

OR

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{mxn}$$

2) Equal Matrices:

Two matrices are said to be equal if there orders and corresponding entries are also equal.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Then A = B

3) Transpose of a matrix:

If we interchange all the rows of a matrix in the respective column or vice versa then new matrix is known as transpose of a matrix. It is denoted by A' or A^T.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

4) Singular matrix:

For a square matrix whose determinant value is zero then it is called a Singular matrix.

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $|A| = 0$

So, A is a singular matrix.

5) Non singular matrix:

For a square matrix whose determinant value is nonzero then it is called a Non singular matrix.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $|A| = -2 \neq 0$

So, A is a Non singular matrix.

7.3 Types of Matrices

1) Row matrix:

A matrix in which there is a single row and any number of columns is called a Row matrix.

$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

2) Column matrix:

A matrix in which there is a single column and any number of rows is called a Column matrix.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3) Zero or Null matrix:

If all the elements of matrix are zero then it is said to be Zero or Null matrix.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

4) Rectangle matrix:

If for a given matrix the number of rows and number of columns are not equal then it is said to be rectangle matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

5) Square matrix:

If for a given matrix the number of rows and number of columns are equal then it is said to be square matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

6) Symmetric matrix:

If for a given square matrix transpose of the matrix is same as the original matrix then it is said to be a Symmetric matrix.

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Here A = A'

7) Skew Symmetric matrix:

If for a given square matrix transpose of the matrix is same as the original matrix but sign is different than it is said to be a Skew – Symmetric matrix.

If $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ then $A' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

Here A = - A'

8) Unit or Identity matrix:

If for a given square matrix in which all the diagonal elements are unity and rest elements are zero then it is said to be a Unit or an Identity matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9) Diagonal matrix:

If for a given square matrix all the elements except the diagonal are zero then it is said to be a Diagonal matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

10) Scalar matrix:

If for a diagonal matrix in which all the diagonal elements are equal then it is said to a scalar matrix.

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

11) Triangular matrix:

If for a square matrix all the elements above or below the principal diagonal are zero then it is said to be a triangular matrix.

Upper Triangular Matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower Triangular Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Note: If all elements below the principal diagonal are zero then it is said to be an upper triangular matrix whereas if all elements above the principal diagonal are zero then it is said to be a lower triangular matrix

12) Orthogonal matrix: If A be a square matrix of order n such that A. $A' = A'$ .A = I (where A' is the transpose of A and I is the identity matrix of order n), then A is called an orthogonal matrix.

e.g.

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$ (It is a proper orthogonal matrix.)

Note:

- If A is an orthogonal matrix then A' is same as inverse of A i.e. $A' = A^{-1}$.
- If A is an orthogonal matrix and if $|A| = 1$ then A is called a proper orthogonal matrix.
- If A is an orthogonal matrix and if $|A| = -1$ then A is called an improper orthogonal matrix.

7.4 Matrix Operation

7.4.1 Addition of a matrix

For given matrix A and B addition is possible if order of the matrices is same and for addition corresponding elements should be added.

7.4.2 Subtraction of a matrix

For given matrix A and B addition is possible if order of the matrices is same and for subtraction corresponding elements should be subtracted.

7.4.3 Scalar Product of a matrix

For given matrix A all the elements of the matrix is multiplied by some constant then the process is known as scalar multiplication.

4.4.4 Multiplication of a matrix

For given matrices A: m x n and B: n x p number of column of the first matrix is same as number of row of the second matrix. So, multiplication exists after multiplying the corresponding row entries of the first matrix with the column entries of second matrix, the resulting matrix will be known as AB: m x p.

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2x2}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}_{2x2}$ then

$$AB = \begin{bmatrix} 1x5 & 2x7 \\ 2x6 & 4x8 \end{bmatrix}_{2x2} = \begin{bmatrix} 5 & 14 \\ 12 & 32 \end{bmatrix}_{2x2}$$

7.4.5 Adjoint matrix of a matrix

Adjoint of matrix:

The transpose of the square matrix obtained by putting the corresponding co – factors at the place of each element of a matrix is called an adjoint of the given matrix and it is denoted by adj. A.

OR

Transpose of a co – factor matrix is known as a transpose of a matrix.

7.4.6 Inverse of a matrix

If A is a square matrix and if there exists another square matrix B of the same order such that AB = AB = I, then B is called inverse of matrix A and it is denoted by A^{-1}.

For non singular matrix A its inverse can be find out by $A^{-1} = \frac{adj.A}{|A|}$.

7.5 Difference between determinants and matrix

No.	Determinant	Matrix
1	In the determinant the number of rows and columns are equal.	In matrix the number of rows and columns are not necessarily equal.
2	In determinants the element are shown between two vertical bars like $\vert\ \vert$.	In matrix the elements are shown in brackets like ( ), { } or [ ].
3	A determinant has a value.	A matrix cannot have a value. It is merely an arrangement.

7.6 Solution of Simultaneous linear equation using inverse matrix

Consider the simultaneous equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

So, it can consider as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Now $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and $|A| \neq 0$ So, A^{-1} exists.

Similarly, it can be extended for any order square matrix.

7.7 Illustrations

1) If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}5 & 1\\ 7 & 3\end{bmatrix}$ then find A + B and A – B.

Solution:

$$A+B=\begin{bmatrix}1+5 & 2+1\\ 3+7 & 4+3\end{bmatrix}=\begin{bmatrix}6 & 3\\ 10 & 7\end{bmatrix}$$

$$A-B=\begin{bmatrix}1-5 & 2-1\\ 3-7 & 4-3\end{bmatrix}=\begin{bmatrix}-4 & 1\\ -4 & 1\end{bmatrix}$$

2) If $A=\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}$ then prove that $A^2 = I$

Solution:

$A^2 = A \text{ x } A$

$$\therefore A^2 = AXA=\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix} X \begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}0+4-3 & 0-12+12 & 0-12+12\\ 0-3+3 & 4+9-12 & 3+9-12\\ 0+4-4 & -4-12+16 & -3-12+16\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

3) Find inverse of the matrix $A=\begin{bmatrix}2 & 3 & 1\\ 0 & 5 & 6\\ 1 & 1 & 2\end{bmatrix}$

Solution:

$$A^{-1}=\frac{adj.A}{|A|}$$

Where, $|A| =$ 2(10 - 6) - 3(0 - 6) + 1(0 - 5) = 21 $\neq$ 0.

So. A^{-1} exists.

Now the co factor matrix is

$$\begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 6 & -5 \\ -5 & 3 & 1 \\ 13 & -12 & 10 \end{bmatrix}$$

After taking transpose of co factor matrix. Adjoin of a matrix is

$$Adj.A = \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{adj.A}{|A|} = \frac{1}{21}\begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

4) By matrix inverse solve the given system of equation
2x + 5y = 16
3x + y = 11

Solution:

So, it can consider as $\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 11 \end{bmatrix}$.

Now $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 16 \\ 11 \end{bmatrix}$ where $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ and $|A| = -13 \neq 0$

So, A^{-1} exists.

$$A^{-1} = \frac{adj.A}{|A|} = -\frac{1}{13}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 16 \\ 11 \end{bmatrix} = -\frac{1}{13}\begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 16 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So, x = 3, y = 2

7.8 Exercise

✓ Theoretical questions

1) Define matrix with proper example.
2) Explain different types of matrices with examples.

3) Give difference between matrix and determinants.
4) Explain inverse of a matrix.
5) Explain addition of the matrices.
6) Explain subtraction of the matrices.
7) Explain multiplication of the matrices.

✓ **MCQs**

1) If A is a square matrix such that A2 = A, then (I – A)3 + A is equal to

a) I
b) 0
c) I – A
d) I + A
Correct option: (a)

2) Diagonal matrix is always
a) Square matrix
b) Rectangle matrix
c) Any type of matrix
d) None of these
Correct option: (a)

3) The given matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is

a) Rectangle matrix
b) Skew symmetric matrix
c) Null matrix
d) Diagonal matrix
Correct option: (d)

4) If A and B are two matrices of the order 3 × m and 3 × n, respectively, and m = n, then the order of matrix (5A – 2B) is

a) m × 3
b) 3 × 3
c) m × n
d) 3 × n
Correct option: (d)

5) Every identity matrix is a diagonal matrix, but the converse is not true.

a) False
b) True
c) Can’t say
d) None of these
Correct option: (b)

6) The matrix $A=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 4\end{bmatrix}$ is a

a) Identity matrix
b) Symmetric matrix
c) Skew symmetric matrix
d) None of these
Correct option: (b)

7) For any two matrices A and B, we have

a) AB = BA
b) $AB \neq BA$
c) AB = O
d) None of the above
Correct option: (d)

8) If A and B are symmetric matrices of the same order, then (AB′ –BA′) is a

a) Skew symmetric matrix
b) Null matrix
c) Symmetric matrix
d) None of these
Correct option: (a)

9) If A is a skew-symmetric matrix, then A^2 is a

a) Skew symmetric matrix
b) Symmetric matrix
c) Null matrix
d) Cannot be determined
Correct option: (b)

10) If $A=\begin{bmatrix}0 & 2\\3 & -4\end{bmatrix}$ and $kA=\begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$, then the values of k, a and b respectively are

a) 6, – 12, – 18
b) 6, – 4, – 9
c) 6, 4, 9
d) 6, 12, 18
Correct option: (b)

✓ **Practical examples**

1) Obtain value of 2A + B – C from the given matrices

$$A=\begin{bmatrix}1 & 4 & 6\\3 & 6 & 4\\2 & 1 & 9\end{bmatrix}, B=\begin{bmatrix}5 & 2 & 2\\4 & 3 & 3\\3 & 2 & 1\end{bmatrix} C=\begin{bmatrix}6 & 10 & 14\\10 & 14 & 11\\7 & 4 & 18\end{bmatrix}$$

Answer: $A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

2) If $A=\begin{bmatrix}6 & 3\\-3 & 9\\12 & -6\end{bmatrix}$ then find matrix B such that 2A' + 3B = 0

Answer: $B=\begin{bmatrix}-4 & 2 & -8\\-2 & -6 & 4\end{bmatrix}$

3) If $A=\begin{bmatrix}2 & 4\\1 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & 3\\0 & 4\end{bmatrix} and C=\begin{bmatrix}1 & 9\\4 & -1\end{bmatrix}$ then prove that A (BC) = (AB) C

4) Prove that $A^2 - 4A - 5I = O$ for the given matrix $A=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$. Where O is a null matrix of order 2 and I is identity matrix of order 2.

5) If $A=\begin{bmatrix}1 & -2\\3 & 4\end{bmatrix}$ then verify that

A . (Adj. A) = (Adj. A) . A = $|A|.I$

Where I is the identity matrix of order 2.

6) Solve the following simultaneous equations by matrix inverse method.

2x +3y = 8

x + 2y = 5

Answer: x = 1, y = 2

7) Solve the following simultaneous equations by matrix inverse method.

x – 3y + z = - 1

2x + y – 4z = -1

6x – 7y + 8z = 7

Answer: x = 1, y = 1, z = 1

UNIT-8 COORDINATE GEOMETRY

8.1 Introduction

"Coordinate geometry is the branch of mathematics which explains problems in geometry with the help of algebra".

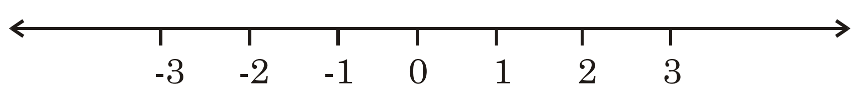
French mathematician René Descartes (1596-1650) introduced this new branch of geometry, also known as Cartesian System. Descartes' coordinate system created a link between algebra and geometry.

In Coordinate geometry, two real numbers, called coordinates are used to indicate the position of a point in a plane. The main contribution of coordinate geometry is that it has enabled the integration of algebra and geometry. This is evident from the fact that algebraic methods are employed to represent and prove the fundamental properties of geometrical theorems. Using coordinate geometry, different function or equation can also be represent in the form of geometric figures. Due to these reasons, coordinate geometry is considered to be more powerful tool of analysis than the Euclidean Geometry.

8.2 Directed Lines

A directed line is a straight line with number units' positive, zero and negative. The point of origin is the number 0.

The arrow indicates its direction.

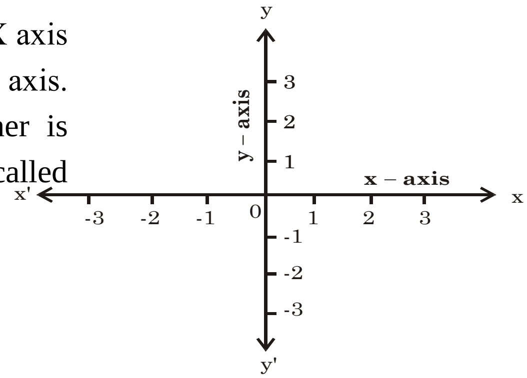


← **Negative Direction** **Origin** **Positive Direction** → **Fig. 8.1**

As per above line, right hand side arrow shows the positive direction and left hand side arrow shows the negative direction. It is like a real number scale.

A Horizontal directed line normally indicated by X’OX axis and vertical directed line normally indicated by Y’OY axis. The point where these two lines intersect each other is called the point of origin. The two lines together are called rectangular axis and are at right angle to each other.

Fig. 8.2



Quadrants :

As per Figure 8.3, two directed lines intersect at right angle at the point of origin and divide their plane into four regions, say XOY, X’O**Y**, X’OY’ and XOY’s.

The region XOY is called first quadrant.

The region X’OY is called second quadrant.

The region X’OY’ is called third quadrant.

The region XOY’ is called fourth quadrant.

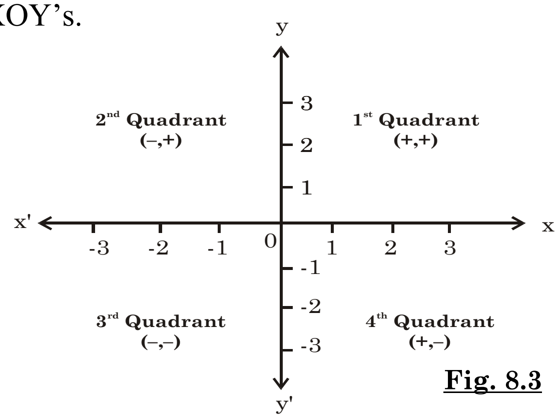
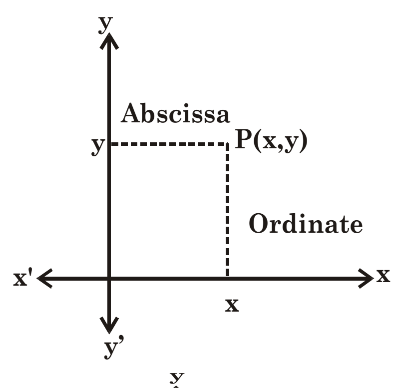


Fig. 8.3

From the Figure 8.3, it is clear that in first quadrant value of x and y is positive, in second quadrant value of x is negative and value of y is positive, in third quadrant value of x and y is negative and in fourth quadrant value of x is positive and y is negative.

Fig. 8.4

In Figure 8.4, P(x, y) be a point on plane. The horizontal distance of the point P from the Y’OY is called the x-coordinate or the abscissa and the vertical distance of the point P from X’OX is called the y-coordinate or the ordinate.



The following coordinates can be plot in the graph (Fig. 8.5).

P(2, 3), Q(–3, 4), R(–2, –3), S(3, –4)

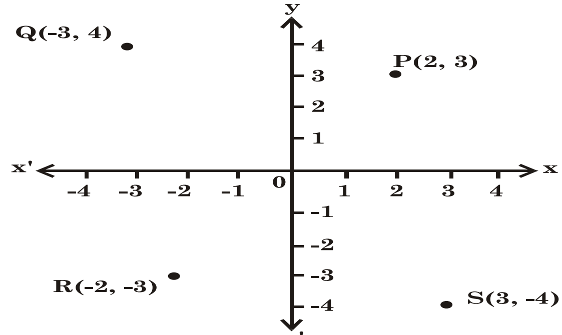


Fig. 8.5

8.3 DISTANCE FORMULA

[1] Distance of a point from the origin :

As shown in Figure 8.6, P(x, y) be a point on plane.

Then by Pythagoras' Theorem,

$OP^2 = OL^2 + PL^2$

$d^2 = x^2 + y^2$

$d = \sqrt{x^2 + y^2}$

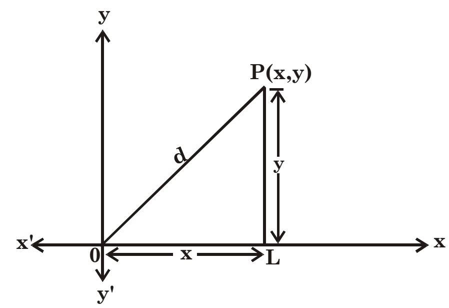


Fig. 8.6

Therefore, distance of a point from the origin O is $d = \sqrt{x^2 + y^2}$.

Ex.1 Find the distance of a point P(4, 3) from the origin.

Solution :

Distance of a point P(4, 3) from the origin is

$OP = d = \sqrt{x^2 + y^2}$ [x = 4, y = 3]

$OP = d = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

d = 5

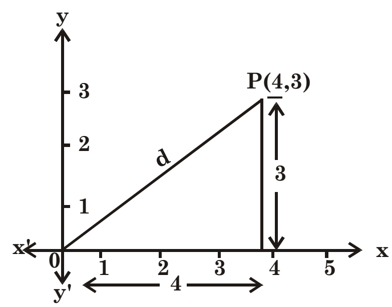


Fig. 8.7

[2] Distance between two points :

As per Figure 8.8, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are any two points on a plane. Then the distance 'd' between P and Q can be obtained as, under.

By Pythagoras' theorem,

$PQ^2 = PT^2 + QT^2$

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

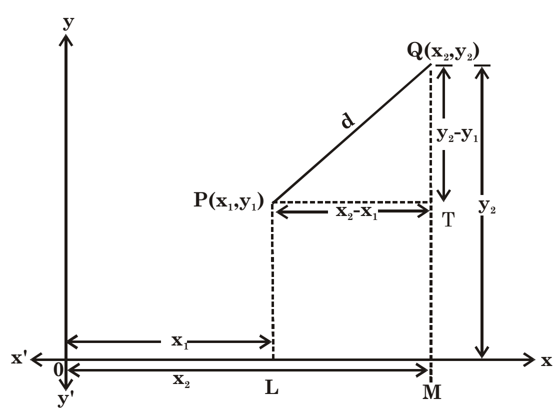


Fig. 8.8

$\therefore$ Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex.2 Find distance between the points (1, 2) and (3, 5)

Solution :

Distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$\therefore$ Distance between two points (1, 2) and (3, 5) is

$$d = \sqrt{(3-1)^2 + (5-2)^2} \qquad [x_1 = 1, y_1 = 2, x_2 = 3, y_2 = 5]$$

$$d = \sqrt{4+9}$$

$$d = \sqrt{13} = 3.5055$$

Ex.3 Find the distance between two points (–1, 4) and (2, 8).

Solution :

Distance between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad [x_1 = -1, y_1 = 4, x_2 = 2, y_2 = 8]$$

$$d = \sqrt{(2-(-1))^2 + (8-4)^2}$$

$$d = \sqrt{9+16} = \sqrt{25} = 5$$

Ex.4 If the point (k, 3) is at a distance of $\sqrt{5}$ units from the point (2, k), then find value of k.

Solution :

Given, distance between two points (k, 3) and (2, k) is $d = \sqrt{5}$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad [x_1 = k, y_1 = 3, x_2 = 2, y_2 = k]$$

$$\therefore \sqrt{5} = \sqrt{(2-k)^2 + (k-3)^2}$$

$5 = (2 - k)^2 + (k - 3)^2$ (by squaring on both side)

$5 = 4 - 4k + k^2 + k^2 - 6k + 9$

$5 = 2k^2 - 10k + 13$

$2k^2 - 10k + 8 = 0$

$k^2 - 5k + 4 = 0$

$k^2 - 4k - k + 4 = 0$

$(k - 1)(k - 4) = 0$

$\therefore$ k = 1 or k = 4.

Ex.5 What will be value of k if the distance between (k, –4) and (–8, 2) is 10?

Solution :

Given, distance between two points (k, –4) and (–8, 2) is d = 10.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [$x_1 = k$, $y_1 = -4$, $x_2 = -8$, $y_2 = 2$]

$10 = \sqrt{(-8-k)^2 + (2-(-4))^2}$

$100 = (-8-k)^2 + (6)^2$ (by squaring on both side)

$100 = 64 + 16\,k + k^2 + 36$

$100 = k^2 + 16k + 100$

$k^2 + 16k = 0$

$k\,(k + 16) = 0,$

$k = 0$ or $k = -16$

Ex.6 Prove that (3, 2), (5, 4), (3, 6), (1, 4) are the vertices of a square.

Solution :

Given $x_1 = 3$, $y_1 = 2$, $x_2 = 5$, $y_2 = 4$, $x_3 = 3$, $y_3 = 6$, $x_4 = 1$, $y_4 = 4$

By using distance formula,

First we find the distance between AB, BC, CD, AD, AC and BD.

By distance formula

$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [$x_1 = 3$, $y_1 = 2$, $x_2 = 5$, $y_2 = 4$]

$d_{AB} = \sqrt{(5-3)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8}$

$d_{BC} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$ [$x_2 = 5$, $y_2 = 4$, $x_3 = 3$, $y_3 = 6$]

$= \sqrt{(3-5)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8}$

$d_{CD} = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$ [$x_3 = 3$, $y_3 = 6$, $x_4 = 1$, $y_4 = 4$]

$= \sqrt{(1-3)^2 + (4-6)^2} = \sqrt{4+4} = \sqrt{8}$

$d_{AD} = \sqrt{(x_1 - x_4)^2 + (y_1 - y_4)^2}$ [$x_1 = 3$, $y_1 = 2$, $x_4 = 1$, $y_4 = 4$]

$= \sqrt{(1-3)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8}$

$$d_{AC} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \qquad [x_1 = 3, y_1 = 2, x_3 = 3, y_3 = 6]$$

$$= \sqrt{(3-3)^2 + (6-2)^2} = \sqrt{0+16} = 4$$

$$d_{BD} = \sqrt{(x_4 - x_2)^2 + (y_4 - y_2)^2} \qquad [x_2 = 5, y_2 = 4, x_4 = 1, y_4 = 4]$$

$$= \sqrt{(1-5)^2 + (4-4)^2} = \sqrt{16+0} = 4$$

From the above result, AB = BC = CD = AD and also AC = BD.

Therefore, given vertices are the vertices of square.

Ex.7 Show that the points (–1, 1), $\left(-\sqrt{3}, -\sqrt{3}\right)$, (1, –1) are the vertices of equilateral triangle.

Solution :

Let P(1, 1), $Q(-\sqrt{3}, -\sqrt{3})$, R(1, 1)

Now, distance between P (–1, 1) and $Q(-\sqrt{3}, -\sqrt{3})$ is d_1,

$$d_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad [x_1 = -1, y_1 = 1, x_2 = -\sqrt{3}, y_2 = -\sqrt{3}]$$

$$d_1 = \sqrt{(-\sqrt{3} - (-1))^2 + (-\sqrt{3} - 1)^2}$$

$$= \sqrt{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1}$$

$$d_1 = \sqrt{8} \qquad \text{..... (1)}$$

Let distance between $Q\left(-\sqrt{3}, -\sqrt{3}\right)$ and R(1, –1) is d_2.

$$d_2 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$d_2 = \sqrt{(1 - (-\sqrt{3}))^2 + (-1 - (-\sqrt{3}))^2} \qquad [x_2 = -\sqrt{3}, y_2 = -\sqrt{3}, x_3 = -1, y_3 = 1]$$

$$= \sqrt{(1 + \sqrt{3})^2 + (-1 + \sqrt{3})^2}$$

$$= \sqrt{1 + 2\sqrt{3} + 3 + 1 - 2\sqrt{3} + 3} = \sqrt{8} \qquad \text{..... (2)}$$

Let distance between R(1, –1) and P (–1, 1) is d_3.

$$d_3 = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2}$$

$d_3 = \sqrt{(-1-1)^2 + (1-(-1))^2}$ [$x_3 = 1$, $y_3 = -1$, $x_1 = -1$, $y_1 =$ 1]

$= \sqrt{(-2)^2 + (2)^2}$

$= \sqrt{4+4}$

$= \sqrt{8}$ ..... (3)

From (1), (2) and (3), PQ = QR = RP.

∴ 3 sides of the triangle are equal.

Hence, given vertices are vertices of equilateral triangle as shown in Figure 8.9.

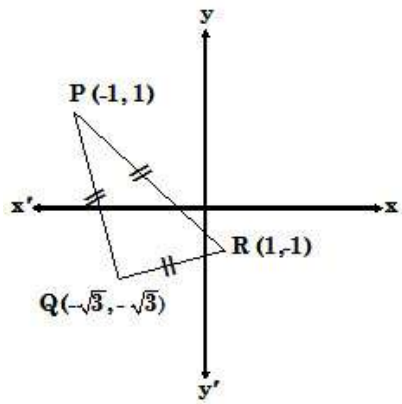


Fig. 8.9

Ex.8 Shows that the triangle whose vertices are (–7, 0), (1, 10) and (2, 1) is an isosceles triangle.

Solution : Let A, B, C be the points (–7, 0), (1, 10) and (2, 1) respectively Given coordinates are $x_1 = -7$, $y_1 = 0$, $x_2 = 1$, $y_2 = 10$, $x_3 = 2$, $y_3 = 1$

$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d_{AB} = \sqrt{(1-(-7))^2 + (10-0)^2}$

$= \sqrt{64+100} = \sqrt{164}$

$d_{BC} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$

$d_{BC} = \sqrt{(2-1)^2 + (1-10)^2}$

$= \sqrt{1+81} = \sqrt{82}$

$d_{AC} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$

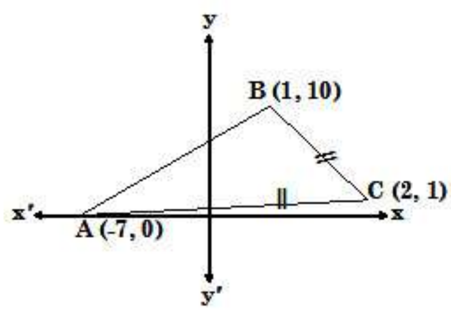


Fig. 8.10

$d_{AC} = \sqrt{(-2-7)^2 + (-1-0)^2} = \sqrt{81+1} = \sqrt{82}$

The triangle whose two sides are equal is called isosceles triangle.

From above result, two of the side, i.e. BC and AC are equal, the triangle is an isosceles triangle.

Ex.9 Show that the set points A (1, –1), B (2, 1) and C (4, 5) are collinear.

Solution :

Given coordinates are $x_1 = 1, y_1 = -1, x_2 = 2, y_2 = 1, x_3 = 4, y_3 = 5$

By using distance formula, first we find the distance between AB, AC and BC.

By distance formula

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d_{AB} = \sqrt{(2-1)^2 + (1-(-1))^2}$$

$$= \sqrt{1+4} = \sqrt{5} \qquad ..... (1)$$

$$d_{BC} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \sqrt{(4-2)^2 + (5-1)^2} = \sqrt{4+16} = \sqrt{20}$$

$$= \sqrt{4 \times 5} = 2\sqrt{5} \qquad ..... (2)$$

$$d_{AC} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$= \sqrt{(4-1)^2 + (5-(-1))^2} = \sqrt{9+36} = \sqrt{45}$$

$$= \sqrt{9 \times 5} = 3\sqrt{5} \qquad ..... (3)$$

From (1), (2) and (3)

$$d_{AB} + d_{BC} = d_{AC}$$

$$\sqrt{5} + 2\sqrt{5} = 3\sqrt{5}$$

$\therefore$ AB + BC = AC

$\therefore$ Given 3 points A, B, C are collinear.

8.4 Section Formula

Internal Division:

As shown in Figure 8.11, the coordinate of a point R (x, y) dividing a line internally in the ratio of $m_1 : m_2$ connecting the points $P(x_1, y_1)$ and Q (x_2, y_2) is given by

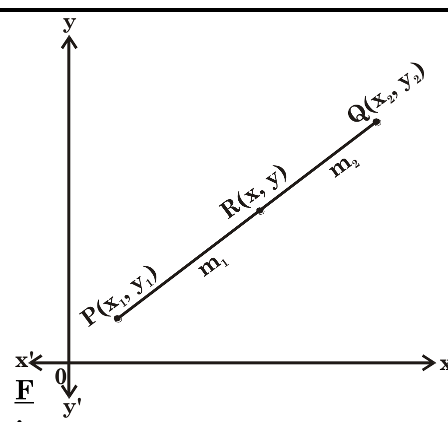


Fig. 8.11

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} \text{ and } y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$$

External Division :

As shown in Figure 8.12, the coordinate of a point R (x, y) dividing a line externally in the ratio of $m_1 : m_2$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by.

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} \text{ and } y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$$

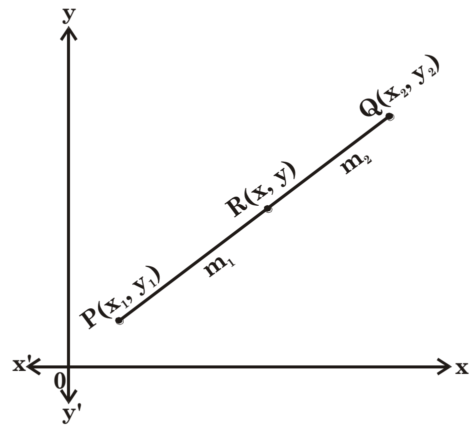


Fig. 8.12

Ex.10 Find the coordinates of the point which divides the points A(–7, 4) and B(8, 9) internally in the ratio 2 : 3.

Solution :

Given coordinates are $x_1 = -7$, $y_1 = 4$ and $x_2 = 8$, $y_2 = 9$.

Internal division in the ratio 2 : 3. $\therefore m_1 = 2$ and $m_2 = 3$.

We substitute the value in the section formula.

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{2(8) + 3(-7)}{2+3} = \frac{16-21}{5} = -\frac{-5}{5} = -1$$

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{2(9) + 3(4)}{2+3} = \frac{18+12}{5} = 6$$

Hence, the coordinate of the points which divides the points A (–7, 4) and B (8, 9) internally in the ratio 2 : 3 is (–1, 6).

Ex.11 Find the coordinate of the point which divides internally the join of the pair of points A(–7, –15) and B(6, –5) in the ratio of 4 : 7.

Solution :

Given coordinates are $x_1 = -7$, $y_1 = -15$ and $x_2 = 6$, $y_2 = -5$

Internal division in the ratio 4 :7. $\therefore$ $m_1 = 4$ and $m_2 = 7$.

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2} = \frac{4(6) + 7(-7)}{4+7} = \frac{24-49}{11} = -\frac{25}{11}$$

$$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2} = \frac{4(-5)+7(-15)}{4+7} = \frac{-20-105}{11} = -\frac{125}{11}$$

Hence, the coordinate of the point which divides the join of the pair of points A(–7, –15) and B(6, –5) in the ratio of 4 : 7 is $\left(-\frac{25}{11}, -\frac{125}{11}\right)$.

Ex.12 Find the coordinate of the point which divides externally the join of the pair of the points A(1, –2) and B(4, 7) in the ratio of 2 : 3.

Solution :

Given coordinate are $x_1=1$, $y_1 = -2$ and $x_2 = 4$, $y_2 = 7$

For external division in the ratio of 2 : 3; $\therefore$ $m_1 = 2$, $m_2 = 3$.

we substitute the above value in the section formula.

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} = \frac{(2)(4)-3(1)}{2-3} = \frac{8-3}{-1} = -5$$

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} = \frac{(2)(7)-(3)(-2)}{2-3} = \frac{14+6}{-1} = -20$$

Hence, coordinate of the point which divides externally the join of the pair of the points A(1, –2) and B(4, 7) in the ratio of 2 : 3 is (–5, –20).

Ex.13 Find the coordinate of the point which divides the points A(–3, 2) and B(4, –3) externally in the ratio of 5 : 2.

Solution :

Given coordinates are $x_1 = -3$, $y_1 = 2$ and $x_2 = 4$, $y_2 = -3$

For external division in the ratio of 5 : 2; $\therefore$ $m_1 = 5$, $m_2 = 2$

We substitute the value in the section formulae.

$$x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2} = \frac{(5)(4)-2(-3)}{5-2} = \frac{20+6}{3} = \frac{26}{3}$$

$$y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2} = \frac{(5)(-3)-2(2)}{5-2} = \frac{-15-4}{3} = -\frac{19}{3}$$

Hence, the coordinate of the point which divides the points A(–3, 2) and B(4, –3) externally in the ratio of 5 : 2 is $\left(\frac{26}{3}, -\frac{19}{3}\right)$.

Ex.14 Find the ratio in which the points C (–1, –1) divides the join of points A (3,3) and B (7, 7).

Solution :

In this example point of division of a line is given and we have to find the ratio of division.

Let the point C (–1, –1) divide the join of points A (3, 3) and B (7, 7) in the ratio m :1.

Given coordinates are $x_1 = 3$, $y_1 = 3$, $x_2 = 7$, $y_2 = 7$

$\therefore$ By section formula, $m_1 = m$, $m_2 = 1$, $x = 1$, $y = -1$

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

$$-1 = \frac{7m + 3(1)}{m + 1}$$

$$-m - 1 = 7m + 3$$

$$-4 = 8m \qquad \therefore\ m = -\frac{1}{2}$$

Here value of m is negative, therefore point C divides the line externally.

$\therefore$ C divides AB externally in the ratio of 1 : 2 . $\left(m : 1 = \frac{1}{2} : 1\right)$

Ex.15 Determine the ratio in which the join of the point A(–2, 3) and B(–4, 6) is divided by the point C(2, –3).

Solution :

Let the point C(2, –3) divide the join of points A(–2, 3) and B(–4, 6) in the ratio m:1.

Given coordinates are $x_1 = -2$, $y_1 = 3$, $x_2 = -4$, $y_2 = 6$, $x = 2$, $y = -3$

$$m_1 = m,\ m_2 = 1$$

By Section formula,

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$

$$2 = \frac{-4m + (1)(-2)}{m + 1}$$

$$2=\frac{-4m-2}{m+1}$$

$$2m+2=-4m-2$$

$$6m=-4$$

$$m=-\frac{4}{6}=-\frac{2}{3}$$

Here value of m is negative, therefore point C divides the line externally.

$\therefore$ C divides AB externally in the ratio $\frac{2}{3}$: 1 or 2 : 3.

Ex.16 In what ratio is the segment joining the pair of points A (2, –4) and B (–3, 6) is divided by the x-axis, also find coordinate on x-axis.

Solution :

Let the point C (x, 0) be on the x-axis, divide the join of points A (2, –4) and B (–3, 6) in the ratio m : 1.

Given coordinates are $x_1 = 2$, $y_1 = -4$, $x_2 = -3$, $y_2 = 6$, $x = x$, $y = 0$

$$m_1 = m, \quad m_2 = 1$$

By Section formula,

$$y=\frac{m_1y_2+m_2y_1}{m_1+m_2}$$

$$0=\frac{(1)(-4)+m(6)}{m+1}$$

$$0=-4+6m$$

$$6m=4$$

$$m=\frac{4}{6}=\frac{2}{3}$$

$\therefore$ C divides AB internally in the ratio $\frac{2}{3}$: 1 or 2 : 3.

By Section formula,

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2}$$

$$x=\frac{2(-3)+3(2)}{2+3}=\frac{-6+6}{5}=0$$

$\therefore$ Given line AB is passing through (0, 0).

8.5 Area Of Triangle

From the Figure 8.13, it is seen that by connecting the point $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, it form triangle. And the area of triangle can be obtained by using the formula.

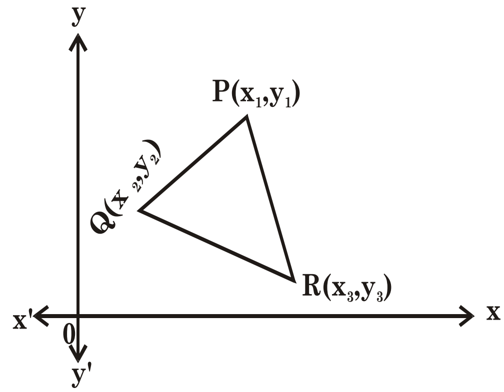


Fig. 8.13

A = Area of Triangle

$$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$A = \frac{1}{2}[x_1 (y_2 - y_3) - x_2 (y_1 - y_3) + x_3 (y_1 - y_2)]$$

$$A = \frac{1}{2}[x_1 y_2 - x_1 y_3 - x_2 y_1 - x_2 y_3 + x_3 y_1 - x_3 y_2]$$

Note : If 3 points are collinear then A = 0.

Ex.17 Find the area of triangle whose vertices are A(2, –1), B (–3, –4) and C (0, 2).

Solution :

Given coordinates are $x_1 = 2$, $y_1 = -1$, $x_2 = -3$, $y_2 = -4$, $x_3 = 0$, $y_3 = 2$. Area of Triangle $= \frac{1}{2}[x_1 (y_2 - y_3) - x_2 (y_1 - y_3) + x_3 (y_1 - y_2)]$

Substituting the given values in above equation we get,

$$A = \frac{1}{2}[2(-4 - 2) - (-3)(-1 - 2) + 0(-1 \;\; (-4))]$$

$$A = \frac{1}{2}[2(-6) + 3(-3) + 0]$$

$$A = \frac{1}{2}(-12 - 9)$$

$$A = -\frac{21}{2}$$

$$|\text{Area of Triangle}| = \frac{21}{2}$$

Ex.18 Find the area of triangle whose vertices are A (x, y – z), B(–x, z) and C(x, y + z).

Solution :

Given coordinates are $x_1 = x$, $y_1 = y - z$, $x_2 = x$, $y_2 = z$, $x_3 = x$, $y_3 = y + z$.

$$\text{Area of Triangle} = \frac{1}{2}\ [x_1 (y_2 - y_3) - x_2 (y_1 - y_3) + x_3 (y_1 - y_2)]$$

$$A = \frac{1}{2}\ [x\ (z - (y + z)\ ) - (-x)\ ((y - z) - (y + z)) + x\ (y - z - z)]$$

$$A = \frac{1}{2} [x(z - y - z) + x\ (y - z - y - z) + x\ (y - 2z)]$$

$$A = \frac{1}{2}\ [x\ (-y) + x\ (-2z) + x\ (y - 2z)]$$

$$A = \frac{1}{2}\ [-xy - 2xz + xy - 2xz]$$

$$A = \frac{1}{2}\ (-4xz)$$

$A = -2xz$ $\quad$ $|\text{Area of Triangle}| = 2xz$

Ex.19 The points A(2, 3/2), B(–3, –7/2), C(k, 9/2) are collinear, then find value of k.

Solution :

Given coordinates are $x_1 = 2$, $y_1 = 3/2$, $x_2 = -3$, $y_2 = -7/2$, $x_3 = k$, $y_3 = 9/2$

3 points are collinear then Area of Triangle = 0

$$\text{Area of Triangle} = \frac{1}{2}\ [x_1 (y_2 - y_3) - x_2 (y_1 - y_3) + x_3 (y_1 - y_2)] = 0$$

$$\frac{1}{2}\left[2\left(-\frac{7}{2}-\frac{9}{2}\right)-(-3)\left(\frac{3}{2}-\frac{9}{2}\right)+k\left(\frac{3}{2}-\left(-\frac{7}{2}\right)\right)\right]=0$$

$$\frac{1}{2}\left[2(-8)+3(-3)+k(5)\right]=0$$

$$-16 - 9 + 5k = 0$$

$$-25 + 5k = 0$$

$$5k = 25$$

$$k = 5$$

8.6 Equation Of Straight Line

"The shortest distance between two distinct points join by a line is called Straight Line."

Equation of straight line is given by

y = mx + c.

Where, m = slope of line

c = intercept on y - axis

From the Figure 8.14, a straight line passing through two points P and Q, intersect y-axis at S(0, c).

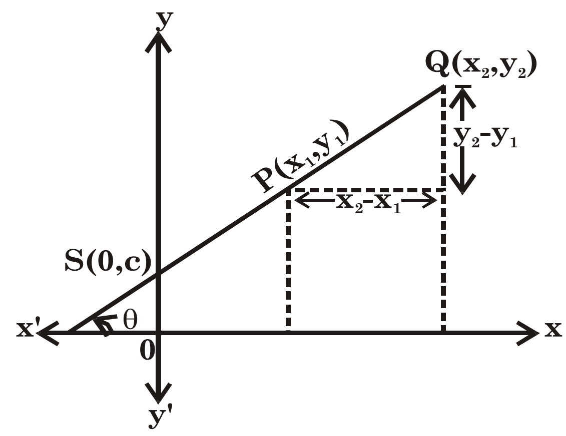


Fig. 8.14

And the slope of line is given by

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1} \text{ or } m = \frac{y_2 - y_1}{x_2 - x_1}$$

(a) As shown in Figure 8.15, if line is parallel to x-axis, then slope of line is zero, i.e.
m = 0 and equation of line is y = c.

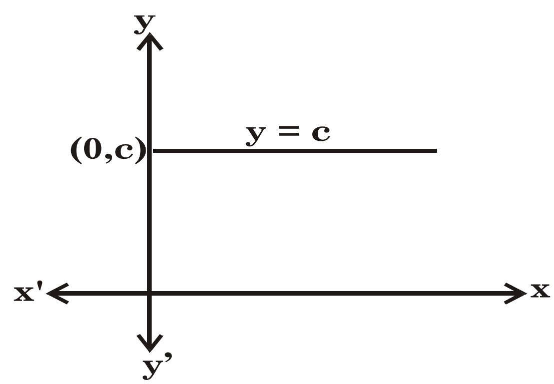


Fig. 8.15

(b) As shown in Figure 8.16, if line is parallel to y-axis, then slope of line is not defined and equation of line is x = a

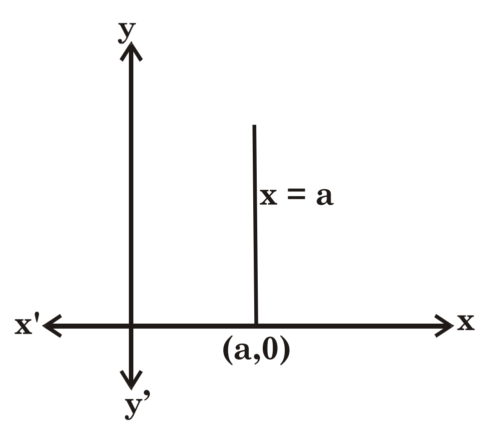


Fig. 8.16

Ex. 20 Find the slope of line passing through the points A (4, 7) and B (1, –2).

Solution :

Given coordinates are $x_1 = 4$, $y_1 = 7$, $x_2 = 1$, $y_2 = -2$.

$$\text{Slope of line} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-2-7}{1-4} = \frac{9}{3} = 3$$

Ex.21 Find the slope of line passing through the points A (7, 5) and B (2, 3).

Solution :

Given coordinates are $x_1 = 7$, $y_1 = 5$, $x_2 = 2$, $y_2 = 3$.

$$\text{Slope of line} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3-5}{2-7} = \frac{2}{5}$$

8.7 General Equation Of Line

[1] ORIGIN SLOPE FORM :

From the Figure 8.17, a straight line is passing through the origin O and have slope m.

Let P (x, y) be any point on the line.

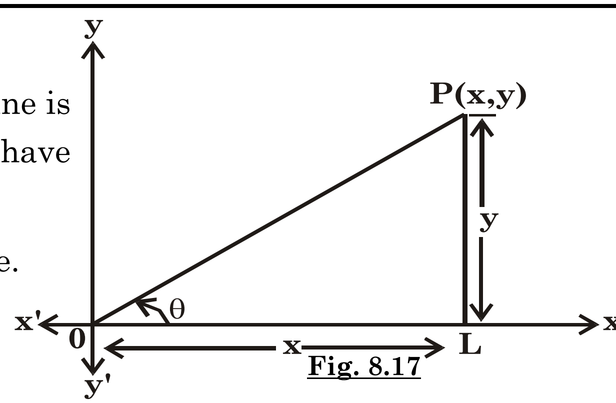


Fig. 8.17

From a point P draw a line PL perpendicular to the x-axis, then

$$\text{Slope of line} = m = \frac{y-0}{x-0} \Rightarrow m = \frac{y}{x} \Rightarrow y = mx$$

which is the required equation of the line.

Ex.22 The equation of straight line passing through the origin and having slope 2 is y = 2x.

[2] A LINE INTERCEPTING THE AXIS:

A straight line which intersect x- axis at any point (other than origin), say, P(a, 0) then point P is called intercept on x-axis.

The straight-line PQ (As shown in Figure 8.18) which intersect the x-axis in P, then OP = a is called the intercept on the x-axis.

Similarly, the straight line which intersect y-axis at any point (other than origin), say Q(0, b), then point Q is called intercept on y-axis.

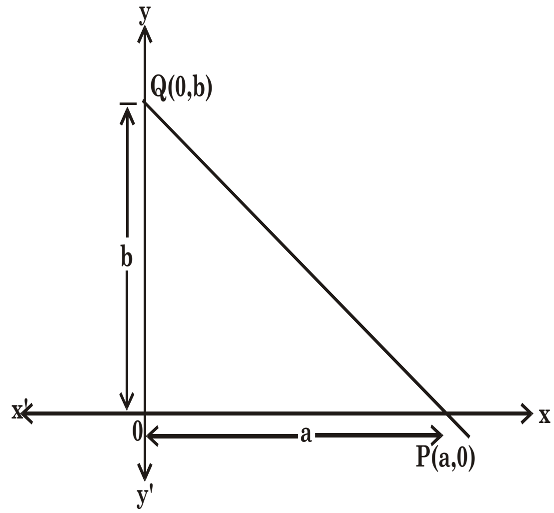


Fig. 8.18

The straight line PQ which intersect the y-axis in Q, then OQ = b is called the intercept on the y-axis.

[3] SLOPE INTERCEPT FORM :

Let a straight line having slope m, intersect the y-axis, in Q (0, c). Let OQ, the intercept on the y-axis be c. Take P(x, y) be any point on the line as shown in Figure 8.19.

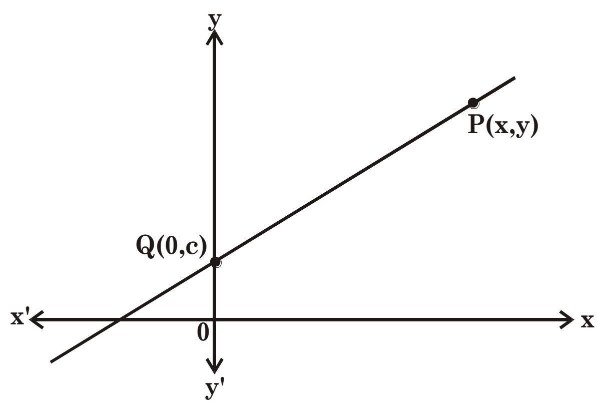


Fig. 8.19

Slope of PQ $= \dfrac{y-c}{x-0}$ ..... (1)

Slope of the line is m. ..... (2)

Therefore, from (1) and (2)

$$m = \frac{y-c}{x-0}$$

$$mx = y - c$$

$$\therefore \quad y = mx + c.$$

Hence, the equation of the line with slope m and an intercept c on y-axis is y = mx + c

Ex.23 Find the equation of a straight line having slope 2 and making intercept on y axis at (0, 2).

Solution :

The equation of the line with the slope m(= 2) and an intercept c(= 2) on y-axis is $y = mx + c$

$y = 2x + 2$ is the required equation of the line.

Ex.24 Find the slope and intercept of the line $2x - 3y = 6$.

Solution :

The equation of straight line having slope m and intercept c on the y-axis is $y = mx + c$.

$2x - 3y = 6$

By arranging the given equation, we get

$3y = 2x - 6$

$$y = \frac{2}{3}x - \frac{6}{3}$$

$$y = \frac{2}{3}x - 2 \qquad (y = mx + c)$$

$\therefore$ Slope of the line = $m = \frac{2}{3}$ and intercept $c = -2$

Ex.25 Find the slope and intercept of the line $6x + 5y - 1 = 0$.

Solution :

The equation of straight line having slope m and intercept c on the y-axis is $y = mx + c$.

$6x + 5y - 1 = 0$

By arranging the given equation, we get

$5y = -6x + 1$

$$y = -\frac{6}{5}x + \frac{1}{5} \qquad (y = mx + c)$$

$\therefore$ Slope of the line = $m = -\frac{6}{5}$ and intercept $c = \frac{1}{5}$.

[4] TWO-INTERCEPT FORM :

Let a straight line intersect the x-axis at P and y-axis at Q (Fig. 8.20), i.e. it makes intercept on x-axis at 'a' and intercept on y-axis at 'b'.

Therefore, the coordinate of P is (a, 0) and Q is (0, b).

Let R (x, y) be any point on the line PQ.

$\therefore$ Slope of PQ = $\dfrac{b-0}{0-a} = -\dfrac{b}{a}$ ..... (1)

Slope of PR $= \dfrac{y-0}{x-a} = \dfrac{y}{x-a}$ ..... (2)

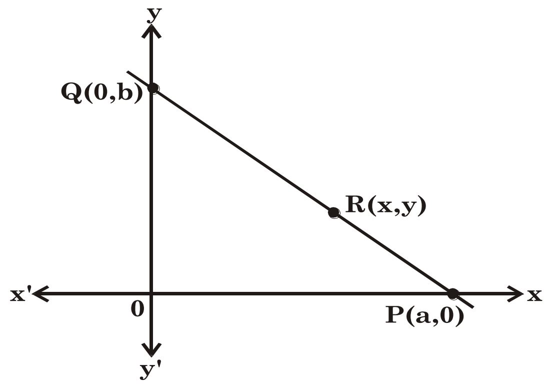


Fig. 8.20

Points P, R, Q are on the same line,

$\therefore$ Slope of PQ = Slope of PR

$$-\frac{b}{a} = \frac{y}{x-a}$$

$$-bx + ab = ay$$

$$bx + ay = ab$$

Dividing both the side by ab we get,

$$\frac{x}{a} + \frac{y}{b} = 1$$

which is the equation of line having intercept '*a*' and '*b*' on the coordinate axis.

Ex.26 Find the equation of straight line having intercept 2 and 3 on the coordinate axis.

Solution :

The equation of line having intercept a(= 2) and b(= 3) on the coordinate axes is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{2} + \frac{y}{3} = 1$$

$$\frac{3x+2y}{6} = 1$$

3x + 2y = 6 is the required equation of the line.

Ex.27 Find the equation of straight line having intercept –1 and 2 on the coordinate axes.

Solution :

The equation of a line having intercept a(= –1) and b(= 2) on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{x}{-1}+\frac{y}{2}=1$$

$$\frac{2x-y}{-2}=1$$

2x – y = –2 is the required equation of the line.

Ex.28 Find the equation of straight line passing through the point (5, 3) and making equal intercept of opposite sign on the axis.

Solution :

Let the line making intercept 'a' and '–a' on the x-axis and y-axis respectively.

Therefore, the equation of the line is,

$$\frac{x}{a}+\frac{y}{-a}=1$$

$$\frac{x-y}{a}=1$$

x – y = a ....(i)

Since the line is passing through (x = 5, y = 3), we have

5 – 3 = a

∴ a = 2

Put a = 2 in (i), we get

∴ The required equation of the straight line is x – y = 2.

Ex.29 Find the equation of straight line passing through the point (3, 4) such that the sum of its intercepts on the axes is 14.

Solution :

Let the line making intercept 'a' and 'b' on the x-axis and y-axis respectively.

Given that, the sum of intercept on the axes is 14, i.e.

$a + b = 14$ ... (i)

Let the equation of the required line be

$$\frac{x}{a}+\frac{y}{b}=1$$

The line passes through (x = 3, y = 4), therefore, we have,

$$\frac{3}{a}+\frac{4}{b}=1$$

$$\frac{3b+4a}{ab}=1$$

$4a + 3b = ab$

$4a + 3(14 - a) = a(14 - a)$ [$\therefore$ From (i) $a + b = 14$ or $b = 14 - a$]

$4a + 42 - 3a = 14a - a^2$

$a + 42 = 14a - a^2$

$a^2 - 13a + 42 = 0$

$(a - 6)(a - 7) = 0$

$\therefore$ $a = 6$ or $a = 7$

When $a = 6$

$b = 14 - a$ or $b = 8$

$\therefore$ The required equations is

$$\frac{x}{6}+\frac{y}{8}=1$$

$$\frac{8x+6y}{48}=1$$

$8x + 6y = 48$

OR

When $a = 7$

$b = 14 - a$ or $b = 7$

$\therefore$ The required equations is

$$\frac{x}{7}+\frac{y}{7}=1 \qquad [\frac{x}{a}+\frac{y}{b}=1]$$

$x + y = 7$

[5] SLOPE POINT FORM

Let the straight line passing through a given point $P(x_1, y_1)$ and make angle θ with x-axis. The slope of the straight line is therefore, $\tan\theta = m$.

Take any point R(x, y) on the straight line as shown in Figure 8.21.

$\therefore$ Slope of PR $= \frac{y - y_1}{x - x_1}$

Since, points Q and R on the same line.

We have,

$\tan \theta = m = \frac{y - y_1}{x - x_1}$

$\therefore \quad y - y_1 = m\,(x - x_1)$

$y = y_1 + m\,(x - x_1)$.

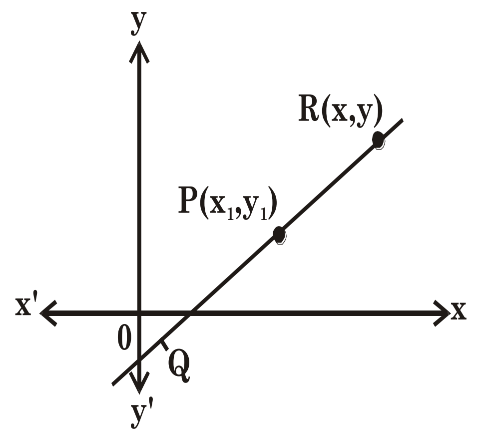


Fig. 8.21

which is the equation of straight line having a slope *m* and passing through the point $P(x_1, y_1)$

Ex.30 Find the equation of straight line having slope 2 and passing through the points (2, 3).

Solution :

Given coordinates are $x_1 = 2$, $y_1 = 3$, $m = 2$.

The equation of the required line is

$y - y_1 = m\,(x - x_1)$

Substituting the given value in above equation,

$y - 3 = 2\,(x - 2)$

$y - 3 = 2x - 4$

$\therefore$ $2x - y = 1$ is the required equation of the line.

Ex.31 Find the equation of straight line passing through the point (–1, 3) and having slope 3.

Solution :

Given coordinates are $x_1 = -1$, $y_1 = 3$, $m = 3$

The equation of required line is

$y - y_1 = m\,(x - x_1)$

Substituting the given value in above equation,

$y - 3 = 3\,(x - (-1))$

$y - 3 = 3\,(x + 1)$

$y - 3 = 3x + 3$

$3x - y = -6$ is the required equation of the line.

[6] TWO POINT FORM

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points on the line.

Take any point R (x, y) on the line as shown in Figure 8.22.

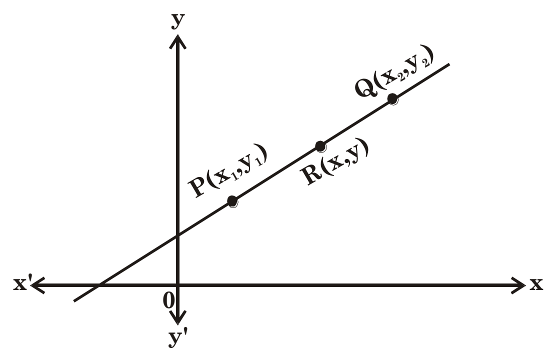


Fig. 8.22

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Let slope of PR $= \frac{y - y_1}{x - x_1}$ ..... (1)

slope of PQ $= \frac{y_2 - y_1}{x_2 - x_1}$ ..... (2)

Since, P, R, Q are collinear (all the three points are on the same line) points, Slope of (1) and (2) are equal.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Which is the equation of straight line passing through two points (x_1, y_1) and (x_2, y_2).

Ex.32 Find the equation of straight line passing through the points A (–1, –1) and B (8, 11)

Solution :

Given coordinates are $x_1 = -1$, $y_1 = -1$, $x_2 = 8$, $y_2 = 11$

The equation of required line is,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - (-1) = \frac{11 - (-1)}{8 - (-1)}(x - (-1))$$

$$y + 1 = \frac{12}{9}(x + 1)$$

$$y + 1 = \frac{4}{3}(x + 1)$$

$$3y + 3 = 4x + 4$$

$$-4 + 3 = 4x - 3y$$

$4x - 3y = -1$ is the required equation of the line.

Ex.33 Find the equation of straight line passing through the points A (–1, 2) and B (2, –1)

Solution :

Given coordinates are $x_1 = -1, y_1 = 2, x_2 = 2, y_2 = -1$

The equation of required line is,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 2 = \frac{-1-2}{2-(-1)}(x-(-1))$$

$$y - 2 = -\frac{3}{3}(x+1)$$

$$y - 2 = -1(x + 1)$$

$x + y = 1$ is the required equation of the line.

Ex.34 The coordinate of two points A and B are (2, 1) and (4, 5) respectively. Find the equation of straight line and the slope of the line AB.

Solution :

Given coordinates are $x_1 = 2, y_1 = 1, x_2 = 4, y_2 = 5$

Slope of line = $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{5-1}{4-2}$$

$$m = 2$$

The equation of required line is,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{5-1}{4-2}(x - 2)$$

$$y - 1 = 2(x - 2)$$

$$y - 1 = 2x - 4$$

$2x - y = 3$ is the required equation of the line.

8.8 Parallel Lines

Let there are two straight lines, i.e., $y = m_1x + c_1$ and $y = m_2x + c_2$. (See Figure 8.23) These two lines are parallel because the angle between them is zero or these two lines will not intersect each other.

Therefore, when two lines are parallel then the slope of both the lines are equal, i.e., $m_1 = m_2$.

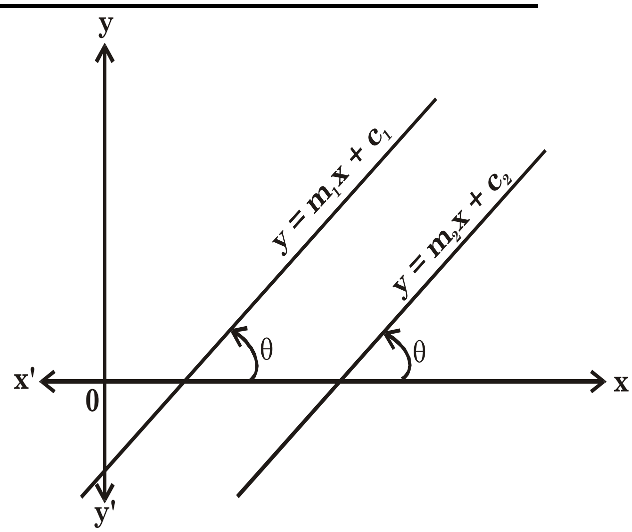


Fig. 8.23

Ex.35 Find the equation of the straight line parallel to $2x - 3y - 5 = 0$ and passing through (4, 5).

Solution :

First, we obtain the slope of line from equation, $2x - 3y - 5 = 0$

$$3y = 2x - 5$$

$$y = \frac{2}{3}x - \frac{5}{3} \quad [y = mx + c]$$

$$\therefore \text{ Slope of line} = m = \frac{2}{3}$$

If two lines are parallel, their slope will be equal.

Therefore, the slope of required line is $m = \frac{2}{3}$.

Hence, the required straight line with slope $m = \frac{2}{3}$ and passing through the point ($x_1 = 4$, $y_1 = 5$) is

$$y - y_1 = m(x - x_1)$$

$$y - 5 = \frac{2}{3}(x - 4)$$

$$3y - 15 = 2x - 8$$

$$8 - 15 = 2x - 3y$$

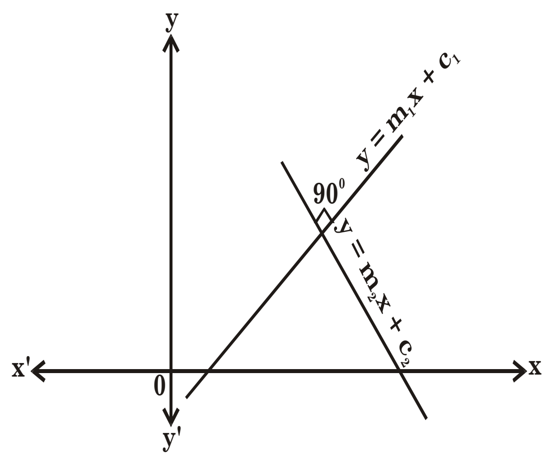
$$2x - 3y = -7$$

8.9 Perpendicular Lines

Let there are two straight lines $y = m_1x + c_1$ and $y = m_2x + c_2$. [See Figure 8.24]

These two lines are perpendicular to each other because the angle between them is $\theta = 90^o$

If two lines are perpendicular, slope of one line is the negative reciprocal of other line



$$\text{i.e., } m_1 = -\frac{1}{m_2} \quad \text{or} \quad m_1 m_2 = -1$$

Fig. 8.24

Therefore, when two lines are perpendicular, $m_1m_2 = -1$.

Ex.36 Find the equation of straight line perpendicular to $5x - 2y + 7 = 0$ and passing through $(-3, 1)$.

Solution :

First, we obtain slope of the line $5x - 2y + 7 = 0$.

$$5x - 2y + 7 = 0$$

$$2y = 5x + 7$$

$$y = \frac{5}{2}x + \frac{7}{2} \qquad (y = m_1x + c_1)$$

$$\text{Slope of line} = m_1 = \frac{5}{2}$$

If two lines are perpendicular, then $m_1 m_2 = -1,\ m_2 = -\frac{1}{m_1}$.

Therefore, the slope of required line is $m_2 = -\frac{1}{5/2} = -\frac{2}{5}$

Hence the required straight line equation having slope ($m_2 = -\frac{2}{5}$) and passing through the point ($x_1 = -3$, $y_1 = 1$) is,

$$y - y_1 = m_2 (x - x_1)$$

$$y - (1) = -\frac{2}{5}(x - (-3))$$

$$y - 1 = -\frac{2}{5}(x + 3)$$

$$5y - 5 = -2x - 6$$

$$2x + 5y = -1.$$

Ex.37 Show that the line joining A(2, 1) and B(3, 4) is perpendicular to the line joining C(7, 5) and D(4, 6).

Solution :

Two lines, AB and CD are perpendicular if,

(slope of AB) (slope of CD) = –1

Let slope of AB = $m_1 = \frac{4-1}{3-2} = 3$ ..... (1) [Slope of line = $\frac{y_2 - y_1}{x_2 - x_1}$]

and slope of CD = $m_2 = \frac{6-5}{4-7} = \frac{1}{-3}$ ..... (2)

From (1) and (2),

$$m_1 m_2 = 3\left(-\frac{1}{3}\right) = -1.$$

If two lines are perpendicular then $m_1 m_2 = -1$

Hence it is proved.

Ex.38 Find the equation of a line parallel to y = 2x – 9 and it passes through the intersection of 5x + y = –4 and 2x + 3y = 1.

Solution :

First, we find the point of intersection from which the line is passing.

To find the point of intersection, using method of elimination we obtain the value of x and y from the given two lines, as :

$$5x + y = -4 \quad ..... (1)$$

$$2x + 3y = 1 \quad ..... (2)$$

By elimination method we get,

$$15x + 3y = -12$$
$$2x + 3y = 1$$
$$- \quad - \quad -$$
$$13x = -13$$
$$x = -1 \text{ and } y = -4 - 5x$$
$$y = -4 - 5(-1)$$
$$y = 1$$

$\therefore$ The point of intersection is (–1, 1).

Now slope of line $y = 2x - 9$ is $m_1 = 2$. [$y = mx + c$]

As the required line is parallel to $y = 2x - 9$, therefore the slope of required line is also 2.

Hence, required equation of straight line passing through the point (–1, 1) and having slope 2 is,

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - (-1))$$

$$y - 1 = 2(x + 1)$$

$$y - 1 = 2x + 2$$

$$2x - y + 3 = 0$$

:: EXERCISE ::

1. What is coordinate geometry? Give the distance formula for the two coordinates (x_1, y_1) and (x_2, y_2).

2. Obtain the equation of the line with slope m and passing through the point (x_1, y_1).

3. Obtain the equation of the line having intercepts 'a' and 'b' on the x-axis and y-axis respectively.

4. Obtain the equation of the line passing through two points (x_1, y_1) and (x_2, y_2).

5. Find the distance between following points.

(i) (0, 0), (4, 3) (ii) (–2, 10), (9, –1) (iii) (–1, 1), (1, –1)

(iv) (3, 3), (6, 4) (v) (5, 2), (6, 4)

6. Show that the points (2, 2), (2, 4), (4, 4) and (4, 2) are the vertices of a square.

7. Show that the points (2, 3), (4, 7) and (6, 6) are the vertices of a right angled triangle.

8. Find the coordinate of the point which divides the points A (8, 9) and B (7, 4) externally in the ratio 4:3

9. Find the coordinate of the point which divides the join of points (3, 4) and (7, 11) externally in the ratio 2:1.

10. Find the coordinate of the point which divides the join of points (2, 1) and (4, 5) internally in the ratio 2:3.

11. Find the ratio in which the point (2, 7) divides the join of points (1, 5) and (3, 9).

12. Find the ratio in which the point (4, 10) divides the join of points (1, 0) and (2, 6).

13. Find the area of triangle whose vertices are

(i) (5, 7), (3, 4), (2, 3) (ii) (0, 0), (1, 2), (1, 2)

14. Prove that the following sets of point are collinear

(i) (1, 3), (3, 5), (–1, 1) (ii) (0, –3) (2, 1) (3, 3) (iii) $\left(-1, -\frac{4}{3}\right), \left(-3, \frac{4}{3}\right), \left(0, -\frac{8}{3}\right)$

15. Find the slope of line, passes through the points

(i) (0, –3), (3, 3) (ii) (1, 4), (3, 8) (iii) (2, 0), (4, –3)

(iv) (3, 2), (7, 5) (v) (1, 4), (3, 6)

16. Find the equation of straight line passes through origin and having slope 3.

17. Find the equation of straight line having slope 1/2 and making intercept on y-axis at (0, 3).

18. Find the equation of straight line having intercept –1 and 2 on the coordinate axis.

19. Find the equation of straight line passes through

(i) The point (–1, 3) and having slope 2.

(ii) The point (2, –3) and having slope –1.

(iii) The point (1/2, 3/2) and having slope 5/2.

20. Find the equation of straight line passes through the points.

(i) (1, 2), (–1, 2) (ii) (1, 5), (3, 9) (iii) (1, 4), (3, 8))

(iv) (1/2, 3/2), (4, 6) (v) (7, 5), (3, 2)

21. Find the equation of straight line passes through the point (2, 4) and making equal intercept of opposite sign on the axis.

22 Find the slope and intercept of the line 2x – 4y = 7.

23. Find the equation of the straight line parallel to y = x + 1 and passing through (3, 4).

24. Find the equation of the straight line parallel to the line joining the points (1, 4) and (2, 6) and passes through (3, 8).

25. Find the equation of straight line perpendicular to 2x – y + 4 = 0 and passes through (3, 2).

26. Prove that the line passing through the points (–3, 5) and (6, –7) is parallel to the line passing through the points (–2, –9) and (–8, –1).

:: ANS. ::

5. (i) 5 (ii) $11\sqrt{2}$ (iii) $2\sqrt{2}$ (iv) $\sqrt{10}$ (v) $\sqrt{5}$

8. (52, 11) **9.** (11, 3) **10.** (14/5, 23/5)

11. (ratio 1:1 internally) **12.** (ratio 5:2 externally) **13.** (i) 23/2 (ii) 2

15. (i) 2 (ii) 2 (iii) –3/2 (iv) 3/4 (v) 1 **16.** y = 3x

17. x – 2y + 6 = 0 **18.** 2x – y + 2 = 0

19. (i) 2x – y + 5 = 0 (ii) x + y + 1 = 0 (iii) 10x – 4y + 1 = 0

20. (i) y = 2, (ii) 2x – y + 3 = 0 (iii) 2x – y + 2 = 0

(iv) 9x – 7y + 6 = 0 (v) 3x – 4y –1 = 0

21. y = x + 2 **22.** slope = 1/2, intercept a = 7/2, b= – 7/4

23. $x - y + 1 = 0$ **24.** $2x - y + 2 = 0$ **25.** $x + 2y - 7 = 0$

➢ **Select the appropriate answer from the given alternative answer. (M.C.Q.)**

1. Distance of a point from the origin is obtained by formula.

(a) $\sqrt{x^2 - y^2}$ (b) $\sqrt{x^2 + y^2}$

(c) $\sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$ (d) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. Distance between two points is obtained by formula.

(a) $\sqrt{x^2 - y^2}$ (b) $\sqrt{x^2 + y^2}$

(c) $\sqrt{(x_2 + x_1)^2 + (y_2 + y_1)^2}$ (d) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

3. The distance of a point P(5, 3) from the origin is

(a) 9 (b) 3 (c) 6 (d) 36

4. The distance of a point P(3, 5) from the origin is

(a) 9 (b) 3 (c) 5.83 (d) 36

5. The coordinate of a point R (x, y) dividing a line internally in the ratio of $m_1 : m_2$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by.

(a) $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$ (b) $x = \frac{m_1x_2 + m_2x_1}{m_1 - m_2}$, $y = \frac{m_1y_2 + m_2y_1}{m_1 - m_2}$

(c) $x = \frac{m_1x_2 - m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 - m_2y_1}{m_1 + m_2}$ (d) $x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}$, $y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$

6. The coordinate of a point R (x, y) dividing a line externally in the ratio of

$m_1 : m_2$ connecting the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by.

(a) $x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$ (b) $x = \frac{m_1x_2 + m_2x_1}{m_1 - m_2}, y = \frac{m_1y_2 + m_2y_1}{m_1 - m_2}$

(c) $x = \frac{m_1x_2 - m_2x_1}{m_1 + m_2}, y = \frac{m_1y_2 - m_2y_1}{m_1 + m_2}$ (d) $x = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, y = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}$

7. The area of triangle whose vertices are (0, 0), (–1, 2), (1, 2) is

(a)1 (b) 2 (c) 3 (d) 4

8. The slope of line, passes through the points (1, 4), (3, 8) is

(a) 1 (b) 2 (c) 3 (d) 4

9. The equation of straight line passes through the origin and having slope 3 is

(a) $y = 3x + 2$ (b) $y = -3x$ (c) $y = 3 - x$ (d) $y = 3x$

10. The equation of straight line passes through the point (–1, 3) and having slope 2 is

(a) $y = -2x + 5$ (b) $y = -5x + 2$ (c) $y = 2x + 5$ (d) $y = 2x + 3$

:: ANS. ::

1. **(b)** 2. **(d)** 3. **(c)** 4. **(c)** 5. **(a)**

6. **(d)** 7. **(b)** 8. **(b)** 9. **(d)** 10. **(c)**

PART – 3

BBA

SEMESTER-1

BUSINESS MATHEMATICS

BLOCK: 3

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Publisher's Name: Dr. Ajaysinh Jadeja
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'JyotirmayParisar',Opp. Shri Balaji
Temple,Chharodi,Ahmedabad,382481,
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Edition: 2024 (First Edition)

ISBN: **978-93-5598-820-1**

978-93-5598-820-1

UNIT-9 DIFFERENTIATION

9.1 Introduction

Calculus is the most important application of mathematics. The present and potential managers of the modern world make extensive uses of this mathematical technique for making important decisions. Calculus is certainly indispensable to measure the degree of changes relating to different managerial issues. Calculus makes it possible for the keen and ambitious executives to determine the relationship of different variables on solid basis. Calculus is concerned with dynamic situations, such as how fast sales levels are increasing, or how rapidly interest is accruing.

The term calculus is primarily related to arithmetic or probability concept. Mathematics determined calculus into two parts - differential calculus and integral calculus. Calculus mainly deals with the rate of changes in a dependent variable with respect to the corresponding change in independent variables. Differential calculus is concerned with the average rate of changes, whereas Integral calculus, by its very nature, considers the total rate of changes in variables.

Differentiation is one of the most important operations in calculus. Its theory solely depends on the concepts of limit and continuity of functions. This operation assumes a small change in the value of dependent variable for small change in the value of independent variable. In fact, the techniques of differentiation of a function deal with the rate at which the dependent variable changes with respect to the independent variable. This rate of change is measured by a quantity known as derivative or differential coefficient of the function. Differentiation is the process of finding out the derivatives of a continuous function i.e., it is the process of finding the differential coefficient of a function.

First definition of derivative was given by Leibnitz (1646 - 1716). Thereafter his competitor Barkley gave another definition. Newton gave correct definition of derivative.

The derivative of a function is its immediate rate of change. Derivative is the small changes in the dependent variable with respect to a very small change in independent variable.

Let $y = f(x)$, derivative i.e. dy/dx means rate of change in variable y with respect to change in variable x. The derivative has many applications, and is extremely useful in optimization- that is, in making quantities as large (for example: profit) or as small (for example: cost) as possible.

Definition 1 :

Let there be a function $y = f(x)$ defined in a certain interval.

Let the argument x receive a certain increment Δx (or h). Then the function y will receive a certain increment Δy so that

$y + \Delta y = f(x + \Delta x)$.

Let us find the increment of the function Δy:

$\Delta y = f(x + \Delta x) - f(x)$.

Hence forming the ratio of increment of the function to the increment of the argument, we get,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Here, the ratio of increment $\frac{\Delta y}{\Delta x}$ is called the **rate of change** of y with respect to x.

We then find the limit of this ratio as $\Delta x \to 0$.

If this limit exists, it is called the derivative of the given function

$y = f(x)$ and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

Thus, by definition,

$$\frac{dy}{dx} \; or \; f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad (\text{Take } h = \Delta x)$$

Thus derivative of given function $y = f(x)$ with respect to x is the limit of the ratio of the increment of the function Δy to the increment of argument Δx, when $\Delta x \to 0$.

Note : Here x is independent variable and y is dependent variable.

Definition 2 :

If $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists, then it is called derivative of the function at $x = a$ and it is denoted by $f'(a)$, i.e. $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

9.2 Differentiation Using First Principle

The operation of finding the derivative of a function $f(x)$ with respect to independent variable x is called differentiation of the function.

The process of finding the differentiation is followed by the following steps:

Step 1 : Put the function to be differentiated equal to y, i.e., $y = f(x)$ ….(i)

Step 2 : Let Δx be an increment in the value of x and Δy, the corresponding increment in the value of y so that,

$$y + \Delta y = f(x + \Delta x) \text{….(ii)}$$

Step 3 : Find Δy by subtracting (i) from (ii), i.e., $\Delta y = f(x + \Delta x) - f(x)$.

Step 4 : Find the ratio $\frac{\Delta y}{\Delta x}$ by dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Step 5 : Take the limits of both sides as $\Delta x \to 0$. This gives us $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ or } \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ (Take } h = \Delta x\text{)}$$

Ex. 1 Find the derivative of x^3 with respect to x with the help of definition.

Solution :

$y = f(x) = x^3$

$y + h = f(x + h) = (x + h)^3$.

Increment of a function $f(x + h)$□$f(x) = (x + h)^3 - x^3$.

Increment ratio $= \frac{(x+h)^3 - (x)^3}{h}$

By definition,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \qquad \left[\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$\frac{dy}{dx} = (3x^2 + 3x(0) + (0)^2)$$

$$\frac{dy}{dx} = 3x^2$$

Ex. 2 Find the derivative of $(x + 1)(x + 2)$ with respect to x using first principle.

Solution :

$f(x) = (x + 1)(x + 2) = x^2 + 3x + 2 \Rightarrow f(x + h) = (x + h)^2 + 3(x + h) + 2$

By definition,

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\left((x+h)^2+3(x+h)+2\right)-\left(x^2+3x+2\right)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{(x^2+2xh+h^2+3x+3h+2)-(x^2+3x+2)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{x^2+2xh+h^2+3x+3h+2-x^2-3x-2}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{2xh+h^2+3h}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{h(2x+h+3)}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0}(2x+h+3) = (2x+0+3) \qquad \frac{dy}{dx} = 2x+3$$

Ex. 3 Find the derivative of the function $f(x) = e^x$ using first principle.

Solution :

Let $y = e^x$ $f(x) = e^x$ $f(x + h) = e^{x + h}$

Now, $f'(x) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$f'(x) = \lim_{h\to 0}\frac{e^{(x+h)}-e^x}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{e^x(e^h-1)}{h}$$

$$f'(x) = \lim_{h\to 0}e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

$$= e^x.1 = e^x$$

:: EXERCISE - 1::

Obtain the derivatives of the following functions with respect to x, by using the definition (first principle method).

1. $f(x) = x + 1$ **2.** $f(x) = x^2 + 1$ **3.** $f(x) = \dfrac{1}{x+2}$ **4.** $f(x) = \dfrac{1}{x}+5$

5. $f(x) = \dfrac{x}{x+9}$ **6.** $f(x) = x^2$□ $5x + 6$ **7.** $f(x) = (x^2 + 1)(x + 3)$

:: ANS. ::

(1) 1 (2) $2x$ (3) $-\frac{1}{(x+2)^2}$ (4) $-\frac{1}{x^2}$

(5) $\frac{9}{(x+9)^2}$ (6) $2x$□ 5 (7) $3x^2 + 6x + 1$

9.3 Differentiation Using Standard Formula

Following are some standard formulas of derivatives by using it, we can easily find the derivatives of algebraic, logarithmic and exponential functions.

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{ax+b}) = ae^x$
3. $\frac{d}{dx}(a^x) = a^x \log_e a$ and $\frac{d}{dx}(a^{bx+c}) = ba^x \log_e a$
4. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
5. $\frac{d}{dx}(k) = 0$, where k is any constant.
6. $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$
7. $\frac{d}{dx}(ku) = k\frac{d}{dx}(u)$, where k is any constant.

Ex. 4 Differentiate the following function with respect to x.

(1) $y = \frac{x^7}{7} - \frac{x^6}{6} - \frac{x^5}{5} - \frac{7}{2}$ (2) $y = \left(x - \frac{1}{x}\right)^2$ (3) $y = \frac{(1-x)^2}{x^2}$

Solution :

(1) $y = \frac{x^7}{7} - \frac{x^6}{6} - \frac{x^5}{5} - \frac{7}{2}$

$$\frac{dy}{dx} = \frac{1}{7}\frac{d}{dx}\left(x^7\right) - \frac{1}{6}\frac{d}{dx}\left(x^6\right) - \frac{1}{5}\frac{dx}{dx}\left(x^5\right) - \frac{d}{dx}\left(\frac{7}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{7}\cdot 7x^6 - \frac{1}{6}6x^5 - \frac{1}{5}5x^4 - 0$$

$$\frac{dy}{dx} = x^6 - x^5 - x^4$$

(2) $y = \left(x - \frac{1}{x}\right)^2$

$$y = x^2 - 2 + \frac{1}{x^2}$$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2) - \frac{d}{dx}(2) + \frac{d}{dx}(x^{-2})$$

$$\frac{dy}{dx} = 2x - 0 - 2x^{-3}$$

$$\frac{dy}{dx} = 2x - \frac{2}{x^3}$$

(3) $y = \dfrac{(1-x)^2}{x^2}$

$$y = \frac{1-2x+x^2}{x^2}$$

$$y = \frac{1}{x^2} - \frac{2x}{x^2} + \frac{x^2}{x^2}$$

$$y = \frac{1}{x^2} - \frac{2}{x} + 1$$

$$y = x^{-2} - 2\,x^{-1} + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) - 2\,\frac{d}{dx}(x^{-1}) + \frac{d}{dx}(1)$$

$$\frac{dy}{dx} = -2(x^{-3}) - 2(-1)\,(x^{-2}) + 0$$

$$\frac{dy}{dx} = -\frac{2}{x^3} + \frac{2}{x^2}$$

Ex. 5 Differentiate the following function with respect to *x*.

(1) $y = \dfrac{\sqrt{x}+2}{\sqrt{x}}$ (2) $y = \dfrac{x^2+2x+1}{x+1}$

Solution :

(1) $y = \dfrac{\sqrt{x}+2}{\sqrt{x}}$

$$y = \frac{\sqrt{x}}{\sqrt{x}} + \frac{2}{\sqrt{x}}$$

$$y = 1 + 2x^{-1/2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(1) + 2\,\frac{d}{dx}(x^{-\frac{1}{2}})$$

$$\frac{dy}{dx} = 0 + 2\,(-\frac{1}{2})(x^{-\frac{3}{2}})$$

$$\frac{dy}{dx} = -x^{-\frac{3}{2}}$$

(2) $y = \frac{x^2 + 2x + 1}{x+1}$

$y = \frac{(x+1)^2}{x+1}$

$y = x + 1$

$\frac{dy}{dx} = 1$

Ex. 6 Differentiate the following function with respect to *x*.

(1) $y = a^x + x^e + e^{-x} + e^e$ (2) $y = \frac{\left(x^2 + x^{-2}\right)\left(x^4 + x^{-4}\right)}{x^3}$

Solution :

(1) $y = a^x + x^e + e^{-x} + e^e$

$\frac{dy}{dx} = \frac{d}{dx}(a^x) + \frac{d}{dx}(x^e) + \frac{d}{dx}(e^{-x}) + \frac{d}{dx}(e^e)$

$\frac{dy}{dx} = a^x \log a + ex^{e-1} + e^{-x}(-1) + (0)$

$\frac{dy}{dx} = a^x \log a + ex^{e-1} - e^{-x}$

(2) $y = \frac{\left(x^2 + x^{-2}\right)\left(x^4 + x^{-4}\right)}{x^3}$

$y = \frac{x^6 + x^{-2} + x^2 + x^{-6}}{x^3}$

$y = \frac{x^6}{x^3} + \frac{x^{-2}}{x^3} + \frac{x^2}{x^3} + \frac{x^{-6}}{x^3}$

$y = x^3 + x^{-5} + x^{-1} + x^{-9}$

$\frac{dy}{dx} = 3x^2 - 5x^{-6} - x^{-2} - 9x^{-10}$

Ex. 7 Differentiate the following function with respect to *x*.

(1) $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\left(x + \frac{1}{x}\right)$ (2) $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 0$

Solution :

(1) $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\left(x + \frac{1}{x}\right)$

$y = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$ $[x^2 - y^2 = (x-y)(x+y)]$

$$y = x^2 - \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) - \frac{d}{dx}(x^{-2})$$

$$\frac{dy}{dx} = 2x - (-2)(x^{-3})$$

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

(2) $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 0$

$y\sqrt{1-x^2} = -x\sqrt{1-y^2}$ (by squaring both sides)

$$y^2(1-x^2) = x^2(1-y^2)$$

$$y^2 - y^2x^2 = x^2 - x^2y^2$$

$$y^2 - x^2 - y^2x^2 + x^2y^2 = 0$$

$$y^2 - x^2 = 0$$

$$y^2 = x^2$$

$$y = x$$

$$\frac{dy}{dx} = 1$$

Ex.8 If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$ then prove that $\frac{dy}{dx} + \frac{x^n}{n!} = y$.

Solution :

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$

$$\frac{dy}{dx} = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + ... + \frac{nx^{n-1}}{n!}$$

$$\frac{dy}{dx} = 1 + \frac{2x}{2!} + \frac{3x^2}{3(2)!} + ... + \frac{nx^{n-1}}{n(n-1)!}$$

$$\frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n-1}}{(n-1)!}$$

by adding $\frac{x^n}{n!}$ both sides,we get

$$\frac{dy}{dx} + \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$$

$$\frac{dy}{dx} + \frac{x^n}{n!} = y$$

9.4 Differentiation of Composite Function

A function is sometimes expressed in terms of one variable.

For example, $(4x^2 + 12x + 7)^3$ is a function of $4x^2 + 12x + 7$, which in turn, is a function of x.

$y = (4x^2 + 12x + 7)^3$.

Take $4x^2 + 12x + 7 = u$.

$\therefore\ y = u^3$

$\therefore$ y is a function of u, and u is a function x.

Thus, y is a function of x.

Chain Rule : If $y = f(u)$, a function of u and $u = g(x)$, a function x, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

For example, $y = (4x^2 + 12x + 7)^3$

Take $u = 4x^2 + 12x + 7$

$\therefore\ y = u^3$

Now, $\frac{dy}{du} = \frac{d}{du}(u^3) = 3u^2$ and

$u = 4x^2 + 12x + 7$

$$\frac{du}{dx} = \frac{d}{dx}(4x^2 + 12x + 7) = 8x + 12$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 3u^2 \cdot (8x + 12)$$

$$\frac{dy}{dx} = 3(4x^2 + 12x + 7)^2 \cdot (8x + 12)$$

We could avoid the step of putting $4x^2 + 12x + 7 = u$ and found the differentiation as under :

$y = (4x^2 + 12x + 7)^3$

$$\frac{dy}{dx} = (4x^2 + 12x + 7)^3$$

$$\frac{dy}{dx} = 3(4x^2 + 12x + 7)^2 \frac{d}{dx}(4x^2 + 12x + 7)$$

$$\frac{dy}{dx} = 3(4x^2 + 12x + 7)^2 \cdot (8x + 12)$$

In practice, we use the above direct process of differentiation of function of function.

Ex.9 Differentiate the following function with respect to x.

(1) $y = (5x^2 + 3x + 2)^4$ (2) $y = e^{4x + 5}$ (3) $y = a^{3x + 2}$

(4) $y = \log(3x + 4)$ (5) $y = (x^2 + 12x + 7)^5$

Solution :

(1) $y = (5x^2 + 3x + 2)^4$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^2 + 3x + 2)^4$$

$$\frac{dy}{dx} = 4(5x^2 + 3x + 2)^3 \frac{d}{dx}(5x^2 + 3x + 2)$$

$$\frac{dy}{dx} = 4(5x^2 + 3x + 2)^3(10x + 3)$$

(2) $y = e^{4x + 5}$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{4x+5})$$

$$\frac{dy}{dx} = e^{4x+5} \frac{d}{dx}(4x + 5)$$

$$\frac{dy}{dx} = e^{4x+5}(4) = 4e^{4x+5}$$

(3) $y = a^{3x + 2}$

$$\frac{dy}{dx} = \frac{d}{dx}(a^{3x+2})$$

$$\frac{dy}{dx} = a^{3x+2} \log a \frac{d}{dx}(3x + 2)$$

$$\frac{dy}{dx} = a^{3x+2} \log a(3) = 3a^{3x+2} \log a$$

(4) $y = \log(3x + 4)$

$$\frac{dy}{dx} = \frac{d}{dx}\log(3x + 4)$$

$$\frac{dy}{dx} = \frac{1}{3x + 4} \frac{d}{dx}(3x + 4)$$

$$\frac{dy}{dx} = \frac{1}{3x + 4}(3) = \frac{3}{3x + 4}$$

(5) $y = (x^2 + 12x + 7)^5$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 12x + 7)^5$$

$$\frac{dy}{dx} = 5(x^2 + 12x + 7)^4 \frac{d}{dx}(x^2 + 12x + 7)$$

$$\frac{dy}{dx} = 5(x^2 + 12x + 7)^4 (2x + 12)$$

Ex.10 Differentiate the following function with respect to *x*.

(1) $y = e^{-x} + a^{2x} + \log x^2$ (2) $y = \log[\log(\log x)]$

Solution :

(1) $y = e^{-x} + a^{2x} + \log x^2$

$y = e^{-x} + a^{2x} + 2\log x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{-x}) + \frac{d}{dx}(a^{2x}) + 2\frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(-x) + a^{2x}\log a\frac{d}{dx}(2x) + 2\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = e^{-x}(-1) + a^{2x}\log a\ (2) + 2\cdot\frac{1}{x}$$

$$\frac{dy}{dx} = -e^{-x} + 2a^{2x}\log a + \frac{2}{x}$$

(2) $y = \log[\log(\log x)]$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\left[\log\left(\log x\right)\right]\right)$$

$$\frac{dy}{dx} = \frac{1}{\log\left(\log x\right)}\cdot\frac{d}{dx}[\log\left(\log x\right)]$$

$$\frac{dy}{dx} = \frac{1}{\log(\log x)}\cdot\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{1}{\log(\log x)}\cdot\frac{1}{\log x}\cdot\frac{1}{x}$$

9.5 Product Rule Of Differentiation

The differential coefficient of the product of two function is the sum of the product of each function with the derivative of other, *i.e.*,

$$\frac{d}{dx}(uv) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}.$$

Proof :

Let $y = uv$ ....(*i*)

Let Δx be a small increment in the value of *x*. Corresponding to the increment in *x*, let there be the increment Δu, Δv, Δy in *u*, *v* and *y* respectively.

Then, $y + \Delta y = (u + \Delta u)(v + \Delta v) = uv + u\Delta v + v\Delta u + \Delta u\,\Delta v$ ....(*ii*)

Subtracting (*i*) from (*ii*), we get

$\Delta y = u\Delta v + v\Delta u + \Delta u\,\Delta v$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

Taking limits as $\Delta x \to 0$ and hence Δu , Δv , Δy all $\to 0$.

$$\lim_{\Delta x \to 0} \frac{\Delta \mathrm{y}}{\Delta \mathrm{x}} = \mathrm{u}\frac{dv}{dx} + \mathrm{v}\frac{du}{dx} + 0\frac{\Delta \mathrm{v}}{\Delta \mathrm{x}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx.}$$

Extension of the rule : If *u*, *v* and *w* are all function of *x*, then

$$\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}$$

Ex.11 Differentiate the following function with respect to *x*.

(1) $y = 2^x \log x$ (2) $y = x^2 \log x$

Solution :

(1) $y = 2^x \log x$

$$\frac{dy}{dx} = \log x \frac{d}{dx}(2^x) + 2^x \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \log x \ (2^x \log 2) + 2^x \frac{1}{x}$$

$$\frac{dy}{dx} = \log x \ (2^x \log 2) + \frac{2^x}{x}$$

(2) $y = x^2 \log x$

$$\frac{dy}{dx} = \log x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \log x(2x) + x^2 \frac{1}{x} = 2x \log x + x$$

Ex.12 Differentiate the following function with respect to *x*.

(1) $y = x\, e^x \log x$ (2) $y = x^2 e^x 4^x$ (3) $y = x^3 e^x \log x$

Solution :

(1) $y = x\, e^x \log x$

$$\frac{dy}{dx} = e^x \log x \frac{d}{dx}(x) + x \log x \frac{d}{dx}(e^x) + xe^x \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = e^x \log x(1) + x \log x(e^x) + xe^x \frac{1}{x}$$

$$\frac{dy}{dx} = e^x \log x + x \log x \ e^x + e^x$$

(2) $y = x^2 e^x 4^x$

$$\frac{dy}{dx} = e^x 4^x \frac{d}{dx}\left(x^2\right) + x^2 4^x \frac{d}{dx}\left(e^x\right) + x^2 e^x \frac{d}{dx}\left(4^x\right)$$

$$\frac{dy}{dx} = e^x 4^x 2x + x^2 4^x e^x + x^2 e^x 4^x \log 4$$

$$\frac{dy}{dx} = x\ e^x 4^x \left(2 + x + x\log 4\right)$$

(3) $y = x^3 e^x \log x$

$$\frac{dy}{dx} = e^x \log x \frac{d}{dx}(x^3) + x^3 \log x \frac{d}{dx}(e^x) + x^3 e^x \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = e^x \log x (3x^2) + x^3 \log x\ e^x + x^3 e^x \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = 3x^2 e^x \log x + x^3 \log x\ e^x + x^2 e^x$$

$$\frac{dy}{dx} = x^2 e^x (3\log x + x\log x + 1)$$

Ex.13 Differentiate the following function with respect to *x*.

(1) $y = x \log x + a\ e^{-x}$ (2) $y = e^{3x}\ \log(5x)$ (3) $y = x^3 \log x + e^x$

Solution:

(1) $y = x \log x + a\ e^{-x}$

$$\frac{dy}{dx} = \log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x) + a \frac{d}{dx}(e^{-x})$$

$$\frac{dy}{dx} = \log x (1) + x \cdot \frac{1}{x} + a(e^{-x}) \frac{d}{dx}(-x)$$

$$\frac{dy}{dx} = \log x + 1 + ae^{-x}(-1)$$

$$\frac{dy}{dx} = \log x + 1 - ae^{-x}$$

(2) $y = e^{3x} \log(5x)$

$$\frac{dy}{dx} = \log(5x) \frac{d}{dx}(e^{3x}) + e^{3x} \frac{d}{dx}(\log(5x))$$

$$\frac{dy}{dx} = \log(5x)\ e^{3x} \frac{d}{dx}(3x) + e^{3x} \frac{1}{5x} \frac{d}{dx}\ (5x)$$

$$\frac{dy}{dx} = \log(5x)\ e^{3x}\ (3) + \frac{e^{3x}}{5x}\ 5$$

$$\frac{dy}{dx} = 3 \log(5x)\ e^{3x} + \frac{e^{3x}}{x} .$$

(3) $y = x^3 \log x + e^x$

$$\frac{dy}{dx} = \log x \frac{d}{dx}\left(x^3\right) + x^3 \frac{d}{dx}\left(\log x\right) + \frac{d}{dx}\left(e^x\right)$$

$$\frac{dy}{dx} = \log x\ 3x^2 + x^3 \cdot \frac{1}{x} + e^x$$

$$\frac{dy}{dx} = 3x^2 \ \log x + x^2 + e^x.$$

9.6 Rule Of Differentiating Quotient Of Two Functions

If *u* and *v* are function of *x* and $v \neq 0$ then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $v \neq 0$

Proof : Let $y = \frac{u}{v}$, ...(*i*)

where *u* and *v* are function of *x*.

Let Δx be a small increment in the value of *x*. Corresponding to the increment Δx in *x*, let there be the increment Δu, Δv, Δy in *u*, *v* and *y* respectively.

Then $y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$ ....(*ii*)

Subtracting (*i*) from (*ii*), we get

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)} = \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

Taking limits as $\Delta x \to 0$ and hence Δu, Δv, Δy all $\to 0$.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v(v + 0)} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex.14 Differentiate the following function with respect to *x*.

(1) $y = \frac{x^2 - 1}{x^2 + 7x}$ (2) $y = \frac{x^2 + 2x + 5}{\log x}$

Solution :

(1) $y = \frac{x^2 - 1}{x^2 + 7x}$ ($u = x^2 - 1$, $v = x^2 + 7x$)

$$\frac{dy}{dx} = \frac{\left(x^2 + 7x\right)\frac{d}{dx}\left(x^2 - 1\right) - \left(x^2 - 1\right)\frac{d}{dx}\left(x^2 + 7x\right)}{\left(x^2 + 7x\right)^2}$$

$$\frac{dy}{dx} = \frac{\left(x^2+7x\right)\left(2x\right)-\left(x^2-1\right)\left(2x+7\right)}{\left(x^2+7x\right)^2}$$

$$\frac{dy}{dx} = \frac{\left(2x^3+14x^2\right)-\left(2x^3+7x^2-2x-7\right)}{\left(x^2+7x\right)^2}$$

$$\frac{dy}{dx} = \frac{2x^3+14x^2-2x^3-7x^2+2x+7}{\left(x^2+7x\right)^2}$$

$$\frac{dy}{dx} = \frac{7x^2+2x+7}{(x^2+7x)^2}$$

(2) $y = \dfrac{x^2 + 2x + 5}{\log x}$

$$\frac{dy}{dx} = \frac{\log x\,\frac{d}{dx}(x^2+2x+5)-(x^2+2x+5)\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x\,(2x+2)-(x^2+2x+5)\left(\frac{1}{x}\right)}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{x\,\log x(2x+2)-(x^2+2x+5)}{x\,(\log x)^2}$$

Ex. 15 Differentiate the following function with respect to *x*.

$$y = \left(1+\frac{1}{x+1}\right)\left(1+\frac{1}{x+2}\right)\left(1+\frac{1}{x+3}\right)$$

Solution :

$$y = \left(1+\frac{1}{x+1}\right)\left(1+\frac{1}{x+2}\right)\left(1+\frac{1}{x+3}\right)$$

$$y = \left(\frac{x+1+1}{x+1}\right)\left(\frac{x+2+1}{x+2}\right)\left(\frac{x+3+1}{x+3}\right)$$

$$y = \left(\frac{x+2}{x+1}\right)\left(\frac{x+3}{x+2}\right)\left(\frac{x+4}{x+3}\right)$$

$$y = \frac{x+4}{x+1}$$

$$\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x+4)-(x+4)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{(x+1)(1)-(x+4)(1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{x+1-x-4}{(x+1)^2}$$

$$\frac{dy}{dx} = -\frac{3}{(x+1)^2}$$

Ex.16 Differentiate the following function with respect to *x*.

(1) $y = \left(\frac{4x+7}{4x-7}\right)^4$ (2) $y = \left(\frac{x}{x+1}\right)^5$

Solution :

(1) $y = \left(\frac{4x+7}{4x-7}\right)^4$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{4x+7}{4x-7}\right)^4$$

$$\frac{dy}{dx} = 4\left(\frac{4x+7}{4x-7}\right)^3 \frac{d}{dx}\left(\frac{4x+7}{4x-7}\right)$$

$$\frac{dy}{dx} = 4\left(\frac{4x+7}{4x-7}\right)^3 \frac{\left[\left(4x-7)\frac{d}{dx}(4x+7)-(4x+7)\frac{d}{dx}(4x-7)\right)\right]}{(4x-7)^2}$$

$$\frac{dy}{dx} = 4\left(\frac{4x+7}{4x-7}\right)^3 \frac{\left[\left(4x-7)(4)-(4x+7)4\right)\right]}{(4x-7)^2}$$

$$\frac{dy}{dx} = 4\left(\frac{4x+7}{4x-7}\right)^3 \frac{\left[\left(16x-28)-(16x+28)\right)\right]}{(4x-7)^2}$$

$$\frac{dy}{dx} = 4\left(\frac{4x+7}{4x-7}\right)^3 \frac{(-56)}{(4x-7)^2}$$

$$\frac{dy}{dx} = -224\,\frac{(4x+7)^3}{(4x-7)^5}$$

(2) $y = \left(\frac{x}{x+1}\right)^5$

$$\frac{dy}{dx} = 5\left(\frac{x}{x+1}\right)^4 \frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$\frac{dy}{dx} = 5\left(\frac{x}{x+1}\right)^4 \left[\frac{(x+1)\frac{d}{dx}(x)-(x)\frac{d}{dx}(x+1)}{(x+1)^2}\right]$$

$$\frac{dy}{dx} = 5\left(\frac{x}{x+1}\right)^4 \left[\frac{(x+1)(1)-(x)(1)}{(x+1)^2}\right]$$

$$\frac{dy}{dx} = 5\frac{x^4}{(x+1)^4}\left[\frac{1}{(x+1)^2}\right] = 5\frac{x^4}{(x+1)^6}$$

9.7 Differentiation Of Logarithmic Function

When a function is expressed in the form of:

(a) $[f(x)]^{g(x)}$, where $f(x)$ and $g(x)$ are functions of x.

(b) A product of number of functions.

(c) A quotient of functions.

Then its derivative can be easily obtained by taking logarithm of first function and then differentiate it.

This method is known as logarithmic differentiation. This concept is more easily understand by given examples herewith.

Ex.17 Differentiate the following function with respect to x.

(1) $y = x^x$ (2) $x^y = e^{x-y}$ (3) $y = x^{\log x}$ (4) $y = x^{e^{-x}}$

Solution :

(1) $y = x^x$

Taking logarithm on both sides, we get

$\log y = \log x^x$

$\log y = x \log x$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(\log y) = \log x \frac{d}{dx}(x) + x\frac{d}{dx}(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \log x(1) + x \cdot \frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \log x + 1$$

$$\frac{dy}{dx} = y\,(\log x + 1)$$

(2) $x^y = e^{x-y}$

Taking logarithm on both sides, we get

$\log x^y = \log (e^{x-y})$

$y \log x = (x - y) \log_e e \quad [\log_e e = 1]$

$y \log x = x - y$

$y \log x + y = x$

$$y\ (\log x + 1) = x$$

$$y = \frac{x}{\log x + 1}$$

Differentiating both sides with respect to *x*, we get

$$\frac{dy}{dx} = \frac{(\log x+1)\frac{d}{dx}(x) - x\frac{d}{dx}(\log x+1)}{(\log x+1)^2}$$

$$\frac{dy}{dx} = \frac{(\log x+1)(1) - x(1/x)}{(\log x+1)^2}$$

$$\frac{dy}{dx} = \frac{\log x + 1 - 1}{(\log x+1)^2} = \frac{\log x}{(\log x+1)^2}$$

(3) $y = x^{\log x}$

Taking logarithm on both sides, we get

$$\log y = \log (x^{\log x})$$

$$\log y = \log x \cdot \log x$$

$$\log y = (\log x)^2$$

Differentiating both sides with respect to *x*, we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\log x)^2$$

$$\frac{1}{y}\frac{dy}{dx} = 2\log x\ \frac{d}{dx}(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\ \log x\ \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{2y}{x}\log x$$

(4) $y = x^{e^{-x}}$

Taking logarithm on both sides, we get

$$\log y = \log x^{e^{-x}}$$

$$\log y = e^{-x}\log x$$

Differentiating both sides with respect to *x*, we get

$$\log y = \log x\frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \log x(-e^{-x}) + e^{-x}\frac{1}{x}$$

$$\frac{dy}{dx} = y\left[-\log xe^{-x} + \frac{e^{-x}}{x}\right]$$

Ex.18 If $x^y y^x = 1$, then find $\frac{dy}{dx}$.

Solution :

$x^y y^x = 1$

Taking logarithm of both sides, we get

$\log (x^y y^x) = \log 1$

$\log x^y + \log y^x = 0$

$y \log x + x \log y = 0$

Differentiating both sides with respect to x, we get

$$y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx} + \log y\frac{d}{dx}(x) + x\frac{d}{dx}(\log y) = 0$$

$$\frac{y}{x} + \log x\frac{dy}{dx} + \log y + \frac{x}{y}\frac{dy}{dx} = 0$$

$$\frac{y}{x} + \log y + \log x\frac{dy}{dx} + \frac{x}{y}\frac{dy}{dx} = 0$$

$$\frac{y}{x} + \log y + \left[\log x + \frac{x}{y}\right]\frac{dy}{dx} = 0$$

$$\frac{y + x\log y}{x} + \left(\frac{y\log x + x}{y}\right)\frac{dy}{dx} = 0$$

$$\left(\frac{y\log x + x}{y}\right)\frac{dy}{dx} = -\frac{y + x\log y}{x}$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)\left(\frac{y + x\log y}{x + y\log x}\right)$$

Ex.19 Differentiate the following function with respect to x.

$$y = \log\left(\frac{1+x^2}{1-x^2}\right)$$

Solution : $y = \log\left(\frac{1+x^2}{1-x^2}\right)$

$$y = \log(1+x^2) - \log(1-x^2)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\log(1+x^2)) - \frac{d}{dx}(\log(1-x^2))$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) - \frac{1}{1-x^2}\frac{d}{dx}(1-x^2)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}(2x) - \frac{1}{1-x^2}(-2x)$$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + \frac{2x}{1-x^2}$$

$$\frac{dy}{dx} = \frac{2x(1-x^2) + 2x(1+x^2)}{(1+x^2)(1-x^2)}$$

$$\frac{dy}{dx} = \frac{4x}{(1-x^4)}$$

9.8 Differentiation Of Implicit Function

Implicit function: If in a given function, neither *x* nor *y* can be expressed in terms of the other variable, then the function is said to be an implicit function otherwise it's called an explicit function.

For example: $x^2 + 4xy^2 + 3y^3 = 0$ is an implicit function.

$y = x^2 + 4x + 5$ is an explicit function.

The derivative of *y* of an implicit function may be obtained as under:

Ex.20 Differentiate the following function with respect to *x*.

(1) $x^3 + 3xy + y^3 = 3$ (2) $x^2 + 2xy + y^2 = \pi$

Solution :

(1) $x^3 + 3xy + y^3 = 3$

$$\frac{d}{dx}(x^3) + 3\left[y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right] + \frac{d}{dx}(y^3) = \frac{d}{dx}(3)$$

$$3x^2 + 3\left[y + x\frac{dy}{dx}\right] + 3y^2\frac{dy}{dx} = 0$$

$$x^2 + y + x\frac{dy}{dx} + y^2\frac{dy}{dx} = 0$$

$$x^2 + y + (x + y^2)\frac{dy}{dx} = 0$$

$$(x + y^2)\frac{dy}{dx} = -(x^2 + y)$$

$$\frac{dy}{dx} = -\frac{(x^2 + y)}{x + y^2}$$

(2) $x^2 + 2xy + y^2 = \pi$

$$\frac{d}{dx}\left(x^2\right) + 2\left[y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right] + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(\pi)$$

$$2x + 2\left[y + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$

$$2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(2x + 2y)\frac{dy}{dx} = -(2x + 2y)$$

$$\frac{dy}{dx} = -\frac{(2x + 2y)}{(2x + 2y)} \qquad \frac{dy}{dx} = -1.$$

Ex.21 If $x = \frac{y+1}{y-1}$ then find $\frac{dy}{dx}$.

Solution :

$$x = \frac{y+1}{y-1}$$

$x\,y - x = y + 1$

$y\,x - y = x + 1$

$y(x - 1) = x + 1$

$$y = \frac{x+1}{x-1}$$

$$\frac{dy}{dx} = \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{x-1-x-1}{(x-1)^2} = -\frac{2}{(x-1)^2}$$

Ex.22 If $x = \frac{1}{1-\frac{1}{y}}$ then find $\frac{dy}{dx}$.

Solution :

$$x = \frac{1}{1-\frac{1}{y}} \Rightarrow x = \frac{1}{\frac{y-1}{y}} \Rightarrow x = \frac{y}{y-1}$$

$xy - x = y$

$xy - y = x$

$y\,(x - 1) = x$

$$y = \frac{x}{x-1}$$

$$\frac{dy}{dx}=\frac{(x-1)\frac{d}{dx}(x)-x\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$\frac{dy}{dx}=\frac{(x-1)(1)-x(1)}{(x-1)^2}$$

$$\frac{dy}{dx}=\frac{x-1-x}{(x-1)^2}$$

$$\frac{dy}{dx}=-\frac{1}{(x-1)^2}$$

9.9 Differentiation Of Parametric Function

If *x* and *y* are both functions of another variable say *t*, i.e., $x = f(t)$, $y = g(t)$, then these equations are called parametric equations. The third variable *t* is called the parameter.

If *x* and *y* are both functions of third variable say *t*, then $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$.

Ex.23 If $x=t^3+t^2+4$ and $y=t^3-2t^2+6$ then find $\frac{dy}{dx}$.

Solution :

Let $y=t^3-2t^2+6$

$$\frac{dy}{dt}=3t^2-4t$$

$$\frac{dy}{dt}=t(3t-4)$$

Now, $x=t^3+t^2+4$

$$\frac{dx}{dt}=3t^2+2t$$

$$\frac{dx}{dt}=t(3t+2)$$

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx}=\frac{t(3t-4)}{t(3t+2)}$$

$$\frac{dy}{dx}=\frac{3t-4}{3t+2}$$

Ex.24 If $x=t^4$ and $y=\frac{t^3}{1+t^2}$ then find $\frac{dy}{dx}$.

Solution :

Let $y = \frac{t^3}{1+t^2}$

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(t^3) - t^3\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)(3t^2) - t^3(2t)}{(1+t^2)^2} \qquad = \frac{3t^2 + 3t^4 - 2t^4}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{3t^2 + t^4}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{t^2(3+t^2)}{(1+t^2)^2}$$

Now, $x = t^4$

$$\frac{dx}{dt} = 4t^3$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{t^2(3+t^2)}{(1+t^2)^2}}{4t^3}$$

$$\frac{dy}{dx} = \frac{(3+t^2)}{4t(1+t^2)^2}$$

Ex.25 If $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+\ t^2}$ then show that, $\frac{dy}{dx} + \frac{x}{y} = 0$

Solution :

$$x = \frac{2t}{1+t^2}$$

$$\frac{dx}{dt} = \frac{\left(1+t^2\right)\frac{d}{dt}\left(2t\right) - 2t\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$= \frac{\left(1+t^2\right)\left(2\right) - 2t\left(2t\right)}{\left(1+t^2\right)^2}$$

$$= \frac{2+2t^2-4t^2}{\left(1+t^2\right)^2}$$

$$= \frac{2-2t^2}{\left(1+t^2\right)^2}$$

$$= \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$y = \frac{1-t^2}{1+t^2}$$

$$\frac{dy}{dt} = \frac{\left(1+t^2\right)\frac{d}{dt}\left(1-t^2\right)-\left(1-t^2\right)\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$= \frac{\left(1+t^2\right)\left(-2t\right)-\left(1-t^2\right)\left(2t\right)}{\left(1+t^2\right)^2}$$

$$= \frac{-2t-2t^3-2t+2t^3}{\left(1+t^2\right)^2}$$

$$= -\frac{4t}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{-4t/\left(1+t^2\right)^2}{2\left(1-t^2\right)/\left(1+t^2\right)^2}.$$

$$= \frac{-4t}{2\left(1-t^2\right)}$$

$$\frac{dy}{dx} = -\frac{2t}{1+t^2}$$

$$\frac{dy}{dx} + \frac{2t}{1-t^2} = 0$$

$$\frac{dy}{dx} + \frac{x}{y} = 0 \qquad \left[\therefore \frac{x}{y} = \frac{2t/\left(1+t^2\right)}{\left(1-t^2\right)/\left(1+t^2\right)} \Rightarrow \frac{x}{y} = \frac{2t}{1-t^2}\right]$$

:: EXERCISE - 2 ::

1. Find $\frac{dy}{dx}$ for the following function :

(a) $y = 4x^2 + 5x + 1$ (b) $y = (3x^2 - 2)(x^2 + 7)$

(c) $y = 7x^7 + 5x^4 - 20x + 37$ (d) $y = 3x^2 - \frac{4}{x^2} + 2^x + 6^2$

2. Find $\frac{dy}{dx}$ for the following function :

(a) $y = (x^2 - 2x + 7)^4$ (b) $y = (x^2 - \log x^2 + e^x)^3$

3. Find $\frac{dy}{dx}$ for the following function :

(a) $y = 2^{x+7}$ (b) $y = 2^{3x^2+4}$ (c) $y = 2^{3x+2^x}$ (d) $y = 2^{\log x + x^2}$

(e) $y = e^{x^2-2x+4}$ (f) $y = e^{2x+\log x}$

4. Find $\frac{dy}{dx}$ for the following function :

(a) $y = x^{11}\log x$ (b) $y = e^x x^2 \log x$ (c) $y = 5^x x^5$ (d) $y = x^3 3^x \log x$

5. Find $\frac{dy}{dx}$ for the following function :

(a) $y = \frac{x^2 - 2x + 7}{x - 1}$ (b) $y = \frac{x^3 - 2x + 4}{x^2 - 1}$

(c) $y = x + \frac{3}{2x + 1}$ (d) $y = \frac{x + \log x}{5 - x}$

6. Find $\frac{dy}{dx}$ for the following function :

(a) $y = \frac{x^2 + 1}{x^2 - 1}$ (b) $y = \frac{x^3}{3^x}$ (c) $y = \log(e^x)$

7. Find $\frac{dy}{dx}$ for the following function :

(a) $x^3 + x^2y + xy^2 + y^3 = 0$ (b) $5x^2 + 12xy + 7y^2 = 10$ (c) $x^3 + y^3 = 3axy$.

8. Find $\frac{dy}{dx}$ for the following function :

(a) $y = \log\left(27^x e^{kx} x^2\right)$ (b) $y = \frac{e^{2x}}{\log x}$

9. If $y\sqrt{1+x} + x\sqrt{1+y} = 0$ and $x \neq y$ show that, $(1+x^2)\frac{dy}{dx} + 1 = 0$.

10. If $x^y = y^x$ then prove that, $\frac{dy}{dx} = \frac{y}{x}\left(\frac{x\log y - y}{y\log x - x}\right)$.

11. If $\sqrt{\frac{y}{x}} + \sqrt{\frac{x}{y}} = 6$ then prove that, $\frac{dy}{dx} = \frac{x - 17y}{17x - y}$.

12. Show that the function defined by the equation $xy - \log y = 1$ satisfies the relationship $y^2 + (xy - 1)\frac{dy}{dx} = 0$

:: ANS. ::

1. **(a)** 8x + 5 **(b)** $12x^3 + 38x$

(c) $49x^6 + 20x^3 - 20$ **(d)** $6x + \frac{8}{x^3} + 2^x \log 2$

2. **(a)** $4(x^2 - 2x + 7)^3(2x - 2)$ **(b)** $3(x^2 - \log x^2 + e^x)^2 (2x - 2/x + e^x)$

3. **(a)** $2^{x+7}\log 2$ **(b)** $6x\,2^{3x^2+4}\log 2$ **(c)** $2^{3x+2^x}\log 2(3 + 2^x \log 2)$

(d) $2^{\log x + x^2}\log 2(\frac{1}{x} + 2x)$ **(e)** $e^{x^2-2x+4}(2x-2)$ **(f)** $e^{2x+\log x}(2+\frac{1}{x})$

4. **(a)** $11x^{10}\log x + x^{10}$ **(b)** $e^x x(x\log x + 2\log x + 1)$

(c) $5^x x^5 \log 5 + 5^{x+1}x^4$ **(d)** $x^2 3^{x+1}\log x + x^3 3^x \log 3 \log x + x^2 3^x$

5. **(a)** $\frac{x^2 - 2x + 5}{(x-1)^2}$ **(b)** $\frac{3x^4 - 2x^3 - x^2 - 8x + 2}{(x^2-1)^2}$

(c) $1 - \frac{6}{(2x+1)^2}$ **(d)** $\frac{4x + 5 + x\log x}{x(5-x)^2}$

6. **(a)** $-\frac{4x}{(x^2-1)^2}$ **(b)** $3x^2 3^{-x} - x^3 3^{-x}\log 3$ **(c)** 1

7. **(a)** $-\frac{x}{y}$ **(b)** $-\frac{(5x+6y)}{(6x+7y)}$ **(c)** $\frac{ay - x^2}{y^2 - ax}$

8. **(a)** $\log 27 + k + \frac{2}{x}$ **(b)** $\frac{e^{2x}(2x\log x - 1)}{x(\log x)^2}$

❖ **Select the appropriate answer from the given alternative answer. (M.C.Q.)**

1. The derivative of $y = \sqrt{x+1}$ is

(a) $1/\sqrt{x+1}$ (b) $-1/\sqrt{x+1}$ (c) $1/2\sqrt{x+1}$ (d) none of these

2. If $f(x) = e^{ax^2+bx+c}$ the $f'(x)$ is

(a) e^{ax^2+bx+c} (b) $e^{ax^2+bx+c}(2ax+b)$ (c) 2ax + b (d) none of these

3. If y = x (x – 1) (x – 2) then $\frac{dy}{dx}$ is

(a) $3x^2 - 6x + 2$ (b) – 6x + 2 (c) $3x^2 + 2$ (d) none of these

4. Given $x = at^2$, y = 2at; $\frac{dy}{dx}$ is calculated as

(a) t (b) – 1 /t (c) 1 /t (d) none of these

5. Given x = 2t + 5, $y = t^2 - 2$; $\frac{dy}{dx}$ is calculated as

(a) t (b) – 1 /t (c) 1 /t (d) none of these

6. If $y = \frac{1}{\sqrt{x}}$ then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2x\sqrt{x}}$ (b) $\frac{-1}{x\sqrt{x}}$ (c) $-\frac{1}{2\sqrt{x}}$ (d) none of these

7. The derivative of $x^2 \log x$ is

(a) 1 + 2log x (b) x(1 + 2 log x) (c) 2 log x (d) none of these

8. The derivative of $\frac{3-5x}{3+5x}$ is

(a) $30 / (3 + 5x)^2$ (b) $1/(3 + 5x)^2$ (c) $-30 / (3 + 5x)^2$ (d) none of these

9. Let $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then $f'(2)$ is equal to

(a) 3/4 (b) 1/2 (c) 0 (d) none of these

10. The derivative of $(x^2 - 1)/x$ is

(a) $1 + 1/x^2$ (b) $1 - 1/x^2$ (c) $1/x^2$ (d) none of these

:: ANS. ::

1. **(c)**	2. **(b)**	3. **(a)**	4. **(c)**	5. **(a)**
6. **(c)**	7.**(b)**	8.**(c)**	9. **(a)**	10. **(a)**

UNIT-10 APPLICATION OF DIFFERENTIATION

10.1 Second And Higher Order Derivative

In previous chapter, we learn that if $y = f(x)$ is the function of x. Then derivative of y is denoted as $\frac{dy}{dx}\left(or\ f'(x)\ or\ y_1\right)$ is the first derivative of y.

The derivative $\frac{dy}{dx}\left(or\ f'(x)\ or\ y_1\right)$ is, in general, another function of x which can be differentiated.

The derivative of $\frac{dy}{dx}\left(\text{or } f'(x) \text{ or } y_1\right)$ is called the second derivative of y and denoted by $\frac{d^2y}{dx^2}\left(\text{or } f''(x) \text{ or } y_2\right)$.

Similarly, the derivative of $\frac{d^2y}{dx^2}\left(\text{or } f''(x) \text{ or } y_2\right)$ is called the third derivative of y and is denoted by $\frac{d^3y}{dx^3}\left(\text{or } f'''(x) \text{ or } y_3\right)$.

In general, the n^{th} derivative of y is denoted by $\frac{d^ny}{dx^n}\left(\text{or } y_n\right)$.

The derivative is a very useful tool for mathematical function, economic function and also other functions related to science to find out the rate of change. In our study higher order derivative is very much useful to find out the maxima and minima of a function, maximum profit, minimum cost, marginal cost, marginal profit, etc.

The applications of the first and higher order derivatives are explained in this chapter.

Ex.1 If $y = x^3 + 2x^2 + 4x$ then find $\frac{d^2y}{dx^2}$.

Solution :

$$y = x^3 + 2x^2 + 4x$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) + 2\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx} = 3x^2 + 4x + 4$$

Differentiating differential equation again with respect to *x* we get,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 3\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) + \frac{d}{dx}(4)$$

$$\therefore \frac{d^2y}{dx^2} = 6x + 4$$

Ex.2 If $y = 6x^4 + 3x^3 + 4x^2 + e^x + 8$ then find $\frac{d^2y}{dx^2}$.

Solution :

$$y = 6x^4 + 3x^3 + 4x^2 + e^x + 8$$

$$\frac{dy}{dx} = 6\frac{d}{dx}(x^4) + 3\frac{d}{dx}(x^3) + 4\frac{d}{dx}(x^2) + \frac{d}{dx}(e^x) + \frac{d}{dx}(8)$$

$$\therefore \frac{dy}{dx} = 24x^3 + 9x^2 + 8x + e^x$$

Differentiating differential equation again with respect to *x* we get,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 24\frac{d}{dx}(x^3) + 9\frac{d}{dx}(x^2) + 8\frac{d}{dx}(x) + \frac{d}{dx}(e^x)$$

$$\therefore \frac{d^2y}{dx^2} = 72x^2 + 18x + 8 + e^x$$

Ex.3 If $y = 3x^4 + 2^x + 4x^2 + e^x + 5\log x$ then find $\frac{d^2y}{dx^2}$.

Solution :

$$y = 3x^4 + 2^x + 4x^2 + e^x + 5\log x$$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x^4) + \frac{d}{dx}(2^x) + 4\frac{d}{dx}(x^2) + \frac{d}{dx}(e^x) + 5\frac{d}{dx}(\log x)$$

$$\therefore \frac{dy}{dx} = 12x^3 + 2^x\log 2 + 8x + e^x + \frac{5}{x}$$

Differentiating differential equation again with respect to *x* we get,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12\frac{d}{dx}(x^3) + \log 2\frac{d}{dx}(2^x) + 8\frac{d}{dx}(x) + \frac{d}{dx}(e^x) + 5\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d^2y}{dx^2} = 36x^2 + (\log 2)(2^x)(\log 2) + 8 + e^x - \frac{5}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} = 36x^2 + (2^x)(\log 2)^2 + 8 + e^x - \frac{5}{x^2}$$

Ex.4 If $f(x) = 2x^4 + 6x^3 + 4x^2 + 5$ then find $f''(2)$.

Solution :

$f(x) = 2x^4 + 6x^3 + 4x^2 + 5$

$$f'(x) = 2\frac{d}{dx}(x^4) + 6\frac{d}{dx}(x^3) + 4\frac{d}{dx}(x^2) + \frac{d}{dx}(5)$$

$\therefore f'(x) = 8x^3 + 18x^2 + 8x$

Differentiating differential equation again with respect to *x* we get,

$$f''(x) = 8\frac{d}{dx}(x^3) + 18\frac{d}{dx}(x^2) + 8\frac{d}{dx}(x)$$

$\therefore\ f''(x) = 24x^2 + 36x + 8$

Now substituting the value of $x = 2$ in above equation we get,

$$\therefore\ f''(2) = 24\left(2^2\right) + 36(2) + 8 = 176$$

10.2 Maxima And Minima Of Function

In chapter 9, we have come across different types of functions which are differentiable. Also we have seen how the higher order derivatives can be found out. Now we discuss the application of derivative in the field of Business and Economics, which are very essential for the interpretation of different types of curves and functions like demand, supply, cost, revenue, production etc..

Mathematical interpretation of marginal, average, and elasticity concept derivative is the only tool. Derivation of their interrelations by using Calculus is possible.

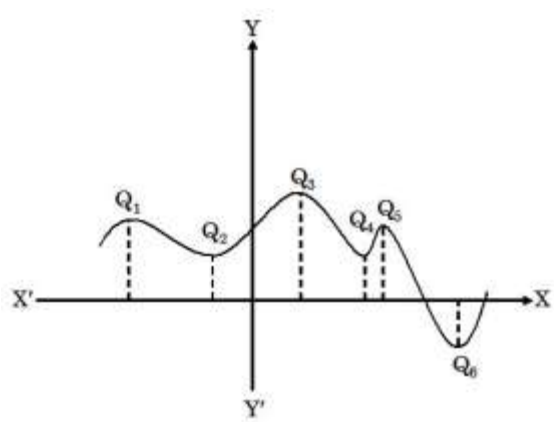
- **Increasing and Decreasing Functions :**

In the function $y = f(x)$, if *y* increases as *x* increase, it is called an increasing function of *x*.

Similarly, if *y* decreases as *x* increases, it is called a decreasing function of *x*.

- **Maxima and Minima :**

It is easy to understand the concept of maxima and minima by plotting the value of function on the graph.



Let us consider the graph of continuous function $y = f(x)$ in the interval (a, b).

Clearly the point Q_1 is the highest in its own immediate neighborhood. Similarly point Q_3.

At each of these points Q_1, Q_3 the function is said to have a maximum value.

On the other hand, the point Q_2 is lowest in its own immediate neighborhood and similarly point Q_4. At each of these points Q_2, Q_4 the function is said to have a minimum value.

Thus, we define maxima and minima of function $y = f(x)$ as under :

Definition: A function $y = f(x)$ is said to have maximum value at $x = a$, if there exist a small number h, however small, such that $f(a) > f(a - h)$ and $f(a + h)$.

A function $y = f(x)$ is said to have minimum value at $x = a$, if there exist a small number h, however small, such that $f(a) < f(a - h)$ and $f(a + h)$.

- **Procedure for finding maxima and minima :**

To find out the maxima and minima of any given function, we follow the following steps :

[1] Let the given function $y = f(x)$.

[2] Find $\frac{dy}{dx}$ or $f'(x)$.

[3] Equate $\frac{dy}{dx}$ or $f'(x)$ to zero.

Solve this equation and find its roots. Let its roots be x_1 , x_2 .

[4] Find $\frac{d^2y}{dx^2}$ or $f''(x)$

[5] Substitute the value of $x = x_1$ in $\frac{d^2y}{dx^2}$ or $f''(x)$.

(1) If $\frac{d^2y}{dx^2}$ or $f''(x_1) > 0$ or positive, $f(x)$ is minima at $x = x_1$.

(2) If $\frac{d^2y}{dx^2}$ or $f''(x_1) < 0$ or negative, $f(x)$ is maxima at $x = x_1$.

(3) If $\frac{d^2y}{dx^2}$ or $f''(x_1) = 0$, $f(x)$ is not maxima or minima at $x = x_1$.

Similarly, substitute the value of $x = x_2$ in second derivative to find the maxima and minima of a function.

Ex.5 Find maximum and minimum value of $y = x^3 + x^2 - 5x + 7$.

Solution : $y = x^3 + x^2 - 5x + 7$

$$\frac{dy}{dx} = 3x^2 + 2x - 5$$

$$\frac{d^2y}{dx^2} = 6x + 2 \qquad \text{........ (i)}$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 2x - 5 = 0$$

$$3x^2 + 5x - 3x - 5 = 0$$

$$(x-1)(3x+5) = 0$$

$$\therefore x = 1 \text{ or } x = -\frac{5}{3}$$

Put $x = 1$ in equation (i), we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 6(1) + 2 = 8 > 0$$

$\therefore$ Given function is minimum at $x = 1$

Minimum $y = x^3 + x^2 - 5x + 7$

Minimum value of y $= (1)^3 + (1)^2 - 5(1) + 7 = 4$

Put $x = 5/3$ in equation (i), we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=-5/3} = 6\left(-\frac{5}{3}\right) + 2 = -8 < 0$$

$\therefore$ Given function is maximum at $x = -\frac{5}{3}$

Maximum $y = x^3 + x^2 - 5x + 7$

Minimum value of y $= (-5/3)^3 + (-5/3)^2 - 5(-5/3) + 7 = 364/27$

Ex.6 Show that the function $f(x) = x^3 - 6x^2 + 12x - 5$ is neither a maximum nor a minimum at $x = 2$.

Solution :

$f(x) = x^3 - 6x^2 + 12x - 5$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6x - 12$ ....(i)

Put $x = 2$ in equation (i), we have

Now, $f''(2) = 6(2) - 12 = 0$

Here, $f''(2) = 0$.

Therefore, given function is neither maximum nor minimum at $x = 2$.

Ex.7 Find the maximum and minimum values of the function

$$f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x + 8$$

Solution : $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$

$f''(x) = 2x^2 + x - 6$

$f''(x) = 4x + 1$ .... (i)

$f'(x) = 0 \Rightarrow 2x^2 + x - 6 = 0$

$2x^2 + 4x - 3x - 6 = 0$

$(2x - 3)(x + 2) = 0$

$\therefore x = \frac{3}{2}$ or $x = -2$

Put $x = 3/2$ in equation (i), we have

$f''(3/2) = 4(3/2) + 1 = 7$

$f''(3/2) > 0$

Here $f''(x) > 0$.

Therefore, given function is minimum at $x = 3/2$.

Minimum $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$

Minimum $f\left(\frac{3}{2}\right) = \frac{2}{3}\left(\frac{3}{2}\right)^3 + \frac{1}{2}\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 8 = \frac{1}{8}$

Put $x = -2$ in equation (i), we have

$f''(-2) = 4(-2) + 1 = -7$

Here, $f''(x) < 0$.

Therefore, given function is maximum at $x = -2$.

Maximum $f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$

Maximum $f(-2) = \frac{2}{3}(-2)^3 + \frac{1}{2}(-2)^2 - 6(-2) + 8 = \frac{38}{3}$

Ex.8 Find the numbers such that $x - y = 100$ and $x^2 - 3y^2$ is maximum.

Solution :

Given, $x - y = 100$ and Maximize $p = (x^2 - 5y^2)$

Let $p = x^2 - 5y^2$

$p = x^2 - 5\,(x - 100)^2$

$p = x^2 - 5\,(x^2 - 200x + 10000)$

$p = x^2 - 5x^2 + 1000x - 50000$

$p = 1000x - 4x^2 - 50000$

$$\frac{dp}{dx} = 1000 - 8x$$

$$\frac{d^2p}{dx^2} = -8$$

$$\frac{dp}{dx} = 0 \Rightarrow 1000 - 8x = 0$$

$$\therefore 1000 = 8x \Rightarrow x = 125$$

$$\frac{d^2p}{dx^2} = -8 < 0$$

$$\therefore p = x^2 - 5y^2 \text{ is Maximum at } x = 125$$

$$\therefore y = x - 100$$

$$\therefore y = 25$$

Ex.9 Find two numbers whose sum is 100 and sum of its square is minimum.

Solution :

Let 2 positive numbers are x and y.

Given sum of 2 numbers are 100.

$x + y = 100$

$y = 100 - x$

Let Sum of squares of 2 number be s.

$s = x^2 + y^2$

$s = x^2 + (100 - x)^2$

$s = x^2 + 10000 - 200x + x^2$

$s = 2x^2 - 200x + 10{,}000$

$$\frac{ds}{dx} = 4x - 200$$

$$\frac{d^2s}{dx^2} = 4$$

$$\frac{ds}{dx} = 0 \Rightarrow 4x - 200 = 0$$

$4x = 200$

$$x = 50$$

$$\frac{d^2s}{dx^2} = 4$$

Here, $\frac{d^2s}{dx^2} > 0,$

Hence, sum of the square of number is minimum when $x = 50$ and $y = 50$.

10.3 Functions Related To Economics

[1] Demand Function :

A relation, which relates the price per unit of a certain product with its quantity demanded *(over a period of time),* is called a ***demand function***. If p is the price per unit of a certain product and x is the number of units of the product which consumers demand during a certain time period, at that price, we can write the demand function as,

$x = f(p)$, showing the dependence of x on p
or $p = g(x)$, showing the dependence *of* p on x.

Generally, an increase in price corresponds to a decrease in the quantity demanded and a decrease in price causes an increase in the quantity demanded.

The graph of a demand function is called **a demand curve** and it is a straight line whose slope is negative, i.e., demand function is a decreasing function. Also negative price and negative quantities are meaningless, so $p > 0$ and $x > 0$.

[2] Supply Function :

A relation which relates the price per unit of a certain product with the quantity of this product supplied by the producers *(over a period of time),* is called *a* ***supply function***. Its graph is called a ***supply curve***.

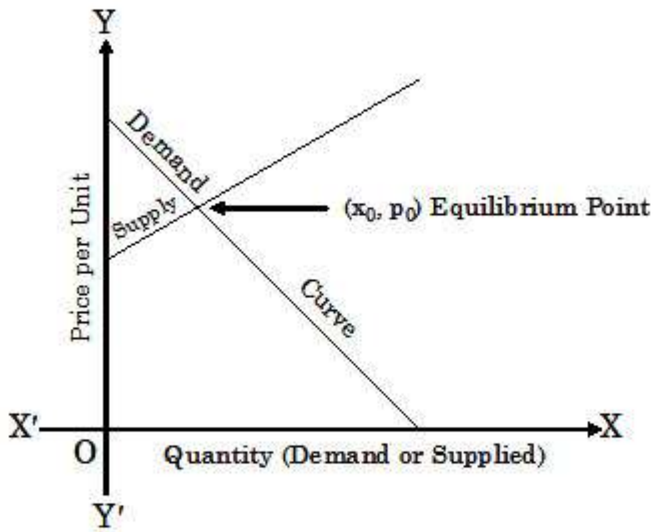
If p denotes the price per unit and x denotes the corresponding quantity supplied of a certain product, then we can write the supply function as

$x = (p)$, showing the dependence of p on x
or $p = (x)$, showing the dependence of x on p.

Generally, an increase in price corresponds to an increase in supply and a decrease in price brings a decrease in supply, meaning that the **supply curve** (which is a straight line) has a positive slope, i.e. supply function is an increasing function. As said in demand function, both p and x are non-negative.

[3] Market Equilibrium (or Market Price) :

General trend in the market is that if price of a certain product is too high, there are lesser number of consumers to purchase it and if its price is too low, then the producers of this commodity do not want to supply it in the market for sale. In the competitive market, when the price per unit of the product depends only on the quantity demanded and the supply available, there is always a tendency for the price to adjust itself in such a manner that the quantity demanded by the consumers, matches with the quantity which the producers, supply willingly at the given price. Such a situation in *Economics* is known as **market equilibrium.**



The point, where the equilibrium exists is called the **equilibrium point,** (x_0, p_0). As shown in above figure. This is the point of intersection of the demand and supply curves. We can obtain it by solving simultaneously the linear equations represented by demand and supply functions.

The **market price** is defined as the price at which the supply and demand for a commodity are equal.

[4] ELASTICITY

Definition : The ratio of the percentage of change in *y* to a given percentage change in *x* is called the elasticity of the function, $y = f(x)$ i.e.

$$\frac{\frac{\Delta y}{y}100}{\frac{\Delta x}{x}100} = \frac{x}{y}\frac{\Delta y}{\Delta x} = \frac{\text{Percentage of change in } y}{\text{Percentage of change in } x}$$

Elasticity of the function is denoted by (eta = η) and obtained as

$$\eta = \frac{x}{y}\frac{dy}{dx}$$

- **Price Elasticity of Demand :**

The degree of responsiveness of the demand for a commodity to a change in its price is called the price elasticity of demand.

If $p = f(x)$ is the demand function (p being the price and x being the quantity demanded) the price elasticity of demand is obtained by

$$\eta_d = -\frac{p}{x}\frac{dx}{dp}$$

The demand is,

(1) elastic when $\eta > 1$ (2) inelastic when $0 < \eta < 1$

(3) perfectly inelastic when $\eta = 0$ (4) unitary when $\eta = 1$

(5) perfectly elastic if $\eta \to \infty$

- **Price Elasticity of Supply :**

The degree of responsiveness of the supply for a commodity to a change in its price is called the price elasticity of supply.

If $p = g(x)$ is the supply function (p being the price and x being the quantity supplied) the price elasticity of supply is obtained by

$$\eta_s = \frac{p}{x}\frac{dx}{dp} = \frac{p_1 + p_o}{p_1 - p_o}\cdot\frac{x_1 - x_o}{x_1 + x_o}$$

Ex.10 The demand curve for a commodity is given by: $x = 15 - \frac{1}{3}p$ (where x represents quantity demanded and p represents the price).

(1) Find the quantity demanded if the price is 6.

(2) Find the price if the quantity demanded is 8.

(3) What is the highest price that would be paid for this commodity?

(4) What quantity would be demanded if the commodity were free?

Solution :

Given : $x = 15 - \frac{1}{3}p$

Let x = quantity demanded and p = price

(1) When $p = 6$: $x = 15 - \frac{1}{3}(6) = 15 - 2 = 13$

(2) When $x = 8$:

$$8 = 15 - \frac{1}{3}p \Rightarrow 8 = \frac{45 - p}{3} \Rightarrow 24 = 45 - p \Rightarrow p = 45 - 24 = 21$$

(3) If x = 0, then $0 = 15 - \frac{1}{3}p \Rightarrow \ 0 = \frac{45 - p}{3} \Rightarrow 0 = 45 - p \Rightarrow p = 45$

This is the highest price.

(4) If the commodity is to be free, p = 0: $x = 15 - \frac{1}{3}(0) = 15$.

This is the quantity demanded if the commodity is free.

Ex.11 Suppose there is demand of 40 units of a product when its price is Rs. 12 per unit and 25 units when its price is Rs. 18 each. Find the demand function, assuming that it is linear. Also determine the price per unit, when 30 units are demanded.

Solution :

Let the demand function be

$p = ax + b$ ....(1)

where p, in rupees, is the price per unit and x is the number of units of the product demanded.

By hypothesis,

when x = 40; p = 12 and x = 25, p = 18.

Substituting these values of x and p in (1), we get

12 = 40a + b

18 = 25a + b

By solving above equation, we get

$a = -\frac{2}{5}$ and b = 12 – 40a = $12 - 40\left(-\frac{2}{5}\right) = 28$

The required demand function is $p = \left(-\frac{2}{5}\right)x + 28$

When, x = 30, we have

$$\text{p} = \left(-\frac{2}{5}\right)(30) + 28 = 16$$

When 30 units are demanded, price of the product is Rs. 16.

Ex.12 If the demand and supply functions are respectively D = 55 – 2p and S = 20 + 1.5 p. Find equilibrium price and quantity.

Solution :

Demand = D = 55 – 2p Supply = S = 20 + 1.5 p

Equilibrium price and quantity can be found out as under,

Demand = Supply

55 – 2p = 20 + 1.5 p

55 – 20 = 2p + 1.5 p

3.5p = 35

$p = 10$

$\therefore$ Equilibrium price = 10

$\therefore$ Equilibrium quantity = $55 - 2p$

$= 55 - 2(10)$

$= 55 - 20 = 35$

Ex.13 The demand functions for a commodity is $p = 10 + 5x - 2x^2$. Find the elasticity of demand at $x = 5$.

Solution :

Given, $p = 10 + 5x - 2x^2$

$$\frac{dp}{dx} = 5 - 4x \Rightarrow \frac{dx}{dp} = \frac{1}{5-4x}$$

Elasticity of demand = $\eta_d = -\frac{p}{x}\frac{dx}{dp}$

$$\eta_d = -\frac{\left(10 + 5x - 2x^2\right)}{x}\frac{1}{5-4x}$$

$$\eta_d = -\frac{10 + 5x - 2x^2}{5x - 4x^2}$$

At $x = 5$, the elasticity of demand is,

$$\eta_d = -\frac{10 + 5(5) - 2(5)^2}{5(5) - 4(5)^2} = -\frac{15}{75}$$

Ex.14 When the price of commodity increases from Rs. 10 per Kg. to Rs. 12 per kg. its supply increase from 1200 kg to 1500 kg., calculate the elasticity of supply of commodities.

Solution :

Initial price = p_o = Rs. 10

New price = p_1 = Rs. 12

Initial supply = x_o = 1200 kg.

New supply = x_1 = 1500 kg.

Elasticity of supply = $\frac{p_1 + p_o}{p_1 - p_o} \cdot \frac{x_1 - x_o}{x_1 + x_o}$

$$= \frac{12+10}{12-10} \cdot \frac{1500-1200}{1500+1200}$$

$$= \frac{22}{2} \cdot \frac{300}{2700} \qquad = \frac{11}{9}$$

10.4 Functions Related To Business

[1] Cost Function :

Let C be the total cost incurred in the production of x units of a commodity. Then, a function relating C and x is called a *cost function* and it is generally written as

$$C = C(x)$$

The cost function consists of two parts:

(1) Fixed Cost : This is denoted by $F(x)$ and it is the sum of all the costs that are independent of the level of production, namely, rent, insurance, depreciation and other overhead expenses, interest on loans and upkeep of the establishment which exist even while there is no production.

(2) Variable Cost : This is denoted by $V(x)$ and it is the sum of all costs that are dependent on the level of production, namely, cost of material, labour, cost of publicity, fuel etc. It depends upon the number, x, of units produced. Thus the total cost function is given by

$$C(x) = V(x) + F(x).$$

(3) Average cost : It is obtained by dividing the total cost by the quantity produced, *i.e.,*

$$\text{Average Cost} = \frac{\text{Total Cost}}{\text{Total Quantity}} \quad \text{or} \quad AC = \frac{C(x)}{x}$$

(4) Marginal Cost : The approximate cost of one additional unit of output is called marginal cost. We can also define marginal cost as the rate of change of total cost, $C(x)$ with respect to x.

First derivative of Cost Function will gives us Marginal Cost Functio

$$MC = \frac{dC}{dx}$$

[2] Revenue Function :

Let R be the total revenue (or the income) of a company which it gets by selling x units of a product at a price p per unit. Then R is given by the relation:

$$R = (\text{price})(\text{quantity sold}) \text{ or } R = p\,x.$$

R is called the total revenue function. It is generally written as $R(x)$.

From the above two definitions, it may be noted that

Average Revenue : It is obtained by dividing the total revenue by the quantity demanded or sold.

$$\text{Average Revenue} = \frac{\text{Total Revenue}}{\text{Total Quantity Sold}} \quad \text{or} \quad AR = \frac{R(x)}{x} \quad \text{or} \quad AR = \frac{p\,x}{x} = p$$

Marginal Revenue : The approximate revenue received from selling of one additional unit of output is called Marginal Revenue. We can also define marginal revenue as the rate of change of total revenue, $R(x)$ with respect to x.

First derivative of Revenue Function will give us Marginal Revenue Function.

$$MR = \frac{dR}{dx}$$

[3] Profit Function :

We know that profit is the excess of revenue over the cost of production. Thus profit function, to be written as $P(x)$, is given by

$$P(x) = R(x) - C(x),$$

where $R(x)$ is the total revenue (income) received by selling x units and $C(x)$ is the total cost incurred in the production of x units of a product.

- **Break-Even Point :**

Revenue earned by the firm is equal to the total cost incurred,

i.e., when $R(x) = C(x)$.

In this case

$$P(x) = R(x) - C(x) = 0.$$

Thus the firm, is neither earning profit nor is it in loss. This is a special case of interest in business and economics. Such a situation is called **business breakeven.** The value of x (*i.e*., volume of production or the number of units produced) when $P(x) = 0$, i.e., when $R(x) = C(x)$ is called **the break-even point.** For this value of x, business is neither in profit nor in loss.

Ex.15 A manufacture of fluorescent tubes has fixed costs of Rs. 2,000 and variable cost of Rs. 15 per unit. Find the equation relating to costs of production. What is the cost of producing 50 tubes? Find the average cost per tube.

Solution :

Here, fixed cost = Rs. 2,000.

Let the number of tubes manufactured per day be x.

Variable cost = Rs. 15 x

$C(x)$ = Cost Function = Fixed cost + Variable cost

$= 2{,}000 + 15x$.

When, $x = 50$,

$C(100) = 2{,}000 + 15(50) = \text{Rs. } 2{,}750.$

$$\therefore \quad \text{Average Cost per Tube} = \frac{C(50)}{50} = \frac{2750}{50} = 55$$

Ex.16 A company manufactures milk powder tins and the cost of manufacture of each tin is Rs. 15 plus a fixed daily overhead cost of Rs. 900. It sells each tin at Rs. 30.

Determine (*a*) cost function, *(b)* revenue function, (*c*) profit function, *(d)* how do you interpret the situation if the company manufactures and sells 400 tins a day?

Solution :

Let x tins be manufactured and sold each day.

(a) Total cost = Fixed cost + Variable cost.
$C(x) = 900 + 15x$

(b) Revenue function = $p\,x$
$R(x)$ = (Sale price per tin)(Number of tins sold) = $30x$.

(c) Profit function = Revenue – Total cost
$P(x) = R(x) - C(x)$
$= 30x - (900 + 15x) = 15x - 900$ ...(1)

(d) Putting $x = 400$ in (1), we get
$P(400) = 15 \times 400 - 900 = 5100$.

The company earns a profit of Rs. 5100 per day by producing and selling 400 tins.

Ex.17 The cost function C of manufacturing a certain article is given by $C = 2x^2 + \frac{108}{x} + 25$ where x is the number of articles manufacture. Find x for minimum value of C. Also calculate minimum cost.

Solution : $C = 2x^2 + \frac{108}{x} + 25$

$$\frac{dC}{dx} = 4x - \frac{108}{x^2}$$

$$\frac{d^2C}{dx^2} = 4 + \frac{216}{x^3} \quad \text{……….. (i)}$$

$$\frac{dC}{dx} = 0 \Rightarrow 4x - \frac{108}{x^2} = 0$$

$$4x = \frac{108}{x^2}$$

$$x^3 = 108/4$$

$x^3 = 27 \Rightarrow x = 3$ or $x = -3$ ($x = -3$ is not possible)

Put $x = 3$ in equation (i), we have

$$\left(\frac{d^2C}{dx^2}\right)_{x=3} = 4 + \frac{216}{3^3} = 4 + 8 = 12 > 0$$

Here, $\frac{d^2C}{dx^2} > 0$.

Therefore, cost is minimum at $x = 2$.

Minimum cost is, $2(3)^2 + 108/3 + 25 = 79$.

Ex.18 A sugar factory produces x tons of sugar every month at the cost of Rs. $\frac{x^3}{3} - 5x^2 - 200x$. Determine the production for minimum cost.

Solution : $C = \frac{x^3}{3} - 5x^2 - 200x$

$$\frac{dC}{dx} = x^2 - 10x - 200$$

$$\frac{d^2C}{dx^2} = 2x - 10$$

$$\frac{dC}{dx} = 0 \Rightarrow x^2 - 10x - 200 = 0$$

$x^2 - 20x + 10x - 200 = 0$

$(x + 10)(x - 20) = 0$

$x = -10$ and $x = 20$ ($x = -10$ is not possible)

$$\left(\frac{d^2C}{dx^2}\right)_{x=20} = 2(20) - 10$$

$= 40 - 10 = 30 > 0$

As $\frac{d^2C}{dx^2} > 0$ at $x = 20$.

 Cost is minimum at $x = 20$.

Ex.19 The total revenue function of a commodity is $R = 30x - \frac{x^2}{2}$. Obtain the demand function. Also find the marginal revenue when the price is 20.

Solution :

Revenue = price × quantity

$\therefore R = p \cdot x$

Demand function = $p = \frac{R}{x}$

$$p = \frac{30x - x^2/2}{x} = 30 - \frac{x}{2}$$

Revenue = $R = 30x - \frac{x^2}{2}$

Marginal Revenue =MR = $\frac{dR}{dx} = 30 - x$

When $p = 20$, then $p = 30 - \frac{x}{2}$

$$20 = 30 - \frac{x}{2}$$

$$\frac{x}{2} = 10 \Rightarrow x = 20$$

Put $x = 20$, we have

$MR = 30 - x \quad = 30 - 20 \quad = 10$

$\therefore$ When $p = 20$, $x = 10$, $MR = 10$

Ex.20 The demand function of a commodity is $p = 100 - 10x$. Find the demand which maximises the revenue.

Solution : Demand function = $p = 100 - 10x$

Revenue = price × quantity

$\therefore \quad R = p \cdot x$

$\therefore \quad R = (100 - 10x)\, x$

$\therefore \quad R = 100x - 10x^2$

$$\frac{dR}{dx} = 100 - 20x$$

$$\frac{d^2R}{dx^2} = -20$$

$$\frac{dR}{dx} = 0 \Rightarrow 100 - 20x = 0;\ 20x = 100;\ x = 5$$

$$\frac{d^2R}{dx^2} = -20 < 0$$

Revenue is maximum at $x = 5$.

Maximum Revenue = $100(5) - 10(5)^2 = 250$.

Ex.21 The demand and cost function of a commodity for a monopolist are $p = 40 - x$, $C = 10 + 5x + \frac{1}{4}x^2$. Find the production for maximum profit. Also obtain the price corresponding to it.

Solution : Revenue = price × quantity

$R = p\,x$

$R = (40 - x)\, x$

$R = 40x - x^2$

Given, $C = 10 + 5x + \frac{1}{4}x^2$

Profit = Revenue – cost

$P = R - C$

$P = (40x - x^2) - (10 + 5x + \frac{1}{4}x^2)$

$P = 35x - \frac{5}{4}x^2 - 10$

$$\frac{dP}{dx} = 35 - \frac{5}{2}x$$

$$\frac{d^2P}{dx^2} = -\frac{5}{2}$$

$$\frac{dP}{dx} = 0 \Rightarrow 35 - \frac{5}{2}x = 0$$

$$\frac{5}{2}x = 35,\ x = 14$$

As $\frac{d^2P}{dx^2} < 0$,

Hence profit is maximum when $x = 14$.

$p = 40 - x$

$p = 40 - 14$

$\therefore\ p = 36$

Ex.22 Let cost function $C = \frac{15}{1000}x^2 + 100$ and revenue function $R = 3x$. Find the production rate x that will maximises profit.

Solution : $P = R - C$ (Profit = Revenue – Cost)

$P = 3x - \frac{15}{1000}x^2 - 100$

$$\frac{dP}{dx} = 3 - \frac{30}{1000}x$$

$$\frac{d^2P}{dx^2} = -\frac{30}{1000}$$

$$\frac{dP}{dx} = 0 \Rightarrow 3 - \frac{30}{1000}x = 0$$

$$\frac{3}{100}x = 3$$

$x = 100$

$$\frac{d^2P}{dx^2} = -\frac{30}{1000}$$

Here, $\frac{d^2P}{dx^2} < 0$.

Therefore, given function is maximum at $x = 100$.

Ex.23 A manufacturer of aluminum boxes sells boxes each at Rs. 76. The cost of manufacturing boxes is $C = 205 + 68x + 0.002x^2$. How many boxes should be manufactured to get maximum profit ? Also find maximum profit.

Solution : Let x be a number of box.

Selling price of a box = Rs. 76

$\therefore$ Revenue = $76x$

Cost function = $C = 205 + 68x + 0.002x^2$

Profit = Revenue – Cost

$P = R - C$

$P = 76x - 205 - 68x - 0.002x^2$

$P = 10x - 205 - 0.002x^2$

$$\frac{dP}{dx} = 10 - 0 - 0.004x$$

$$\frac{d^2P}{dx^2} = -0.004$$

$$\frac{dP}{dx} = 0 \Rightarrow 10 - 0.004x = 0$$

$0.004x = 10$

$x = 2500$

As $\frac{d^2P}{dx^2} < 0$

Profit is Maximum at $x = 2500$.

$P = 10x - 205 - 0.002x^2$

Maximum Profit = $10(2500) - 205 - 0.002(2500)^2$

$= 25000 - 205 - 12500 = 12295$

10.5 Consumption Function

If C denotes the total national consumption and I denote the total national income, then the function $C = f(I)$, relating to I and C, is called the consumption function.

The difference between the total income, I and the total consumption, C is saving, S.

Therefore, Saving = Total Income – Total Consumption *i.e.*,

$$S = I - C.$$

[1] Marginal Propensity to Consume (*MPC*) :

The rate of change of consumption with respect to income is called the Marginal Propensity to Consume (*MPC*).

If $C = f(I)$ then $MPC = \frac{dC}{dI}$

[2] Marginal Propensity to Save (*MPS*) :

The rate of change of saving with respect to income is called the Marginal Propensity to save (*MPS*).

If $S = I - C$ then $MPS = \frac{dS}{dI} = 1 - \frac{dC}{dI}$

Ex.24 If the consumption function is given by $C = 4 + 2\sqrt{I}$, determine the marginal propensity to consume and marginal propensity to save when $I = 25$.

Solution : Given, $C = 4 + 2\sqrt{I}$

[1] Marginal Propensity to Consume (*MPC*) :

$$MPC = \frac{dC}{dI}$$

$$\frac{dC}{dI} = \frac{d}{dI}(4) + 2\frac{d}{dI}(\sqrt{I})$$

$$\frac{dC}{dI} = 0 + 2\frac{1}{2\sqrt{I}}$$

$$\frac{dC}{dI} = \frac{1}{\sqrt{I}}$$

Now, *MPC* at $I = 25$ is,

$$\frac{dC}{dI} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

[2] Marginal Propensity to Save (*MPS*) :

$$MPS = \frac{dS}{dI} = 1 - \frac{dC}{dI}$$

$$\frac{dS}{dI} = 1 - \frac{1}{\sqrt{I}}$$

Now, *MPS* at I = 25 is,

$$\frac{dS}{dI} = 1 - \frac{1}{5} = \frac{4}{5}$$

Ex.25 If the consumption function is given by $C = 10 + \frac{3}{5}I - \frac{1}{4}\sqrt{I}$, determine the marginal propensity to consume and marginal propensity to save when I = 16.

Solution : Given, $C = 10 + \frac{3}{5}I - \frac{1}{4}\sqrt{I}$

[1] Marginal Propensity to Consume (*MPC*) :

$$MPC = \frac{dC}{dI}$$

$$\frac{dC}{dI} = \frac{d}{dI}(10) + \frac{3}{5}\frac{d}{dI}(I) - \frac{1}{4}\frac{d}{dI}(\sqrt{I})$$

$$\frac{dC}{dI} = 0 + \frac{3}{5} - \frac{1}{4}\frac{1}{2\sqrt{I}}$$

$$\frac{dC}{dI} = \frac{3}{5} - \frac{1}{8\sqrt{I}}$$

Now, *MPC* at I = 16 is,

$$\frac{dC}{dI} = \frac{3}{5} - \frac{1}{8\sqrt{16}} = \frac{3}{5} - \frac{1}{32} = \frac{91}{160}$$

[2] Marginal Propensity to Save (*MPS*) :

$$MPS = \frac{dS}{dI} = 1 - \frac{dC}{dI}$$

$$\frac{dS}{dI} = 1 - \frac{3}{5} + \frac{1}{8\sqrt{I}}$$

Now, *MPS* at I = 16 is,

$$MPS = \frac{dS}{dI} = 1 - \frac{3}{5} + \frac{1}{8\sqrt{16}}$$

$$\frac{dS}{dI} = 1 - \frac{3}{5} + \frac{1}{32} = \frac{69}{160}$$

:: EXERCISE ::

1. Explain the concept of maxima and minima of one variable function $y = f(x)$ also states its working rules.

2. Find the maximum and minimum value of the function $y = x^3 - 9x^2 + 24x + 2$.

3. Find the maximum and minimum value of the function $f(x) = x^3 - 6x^2 + 9x + 5$.

4. If $f(x) = x^3 - 6x^2 - 9x - 8$, then for what value of x, $f(x)$ is maximum and minimum? Also find maximum and minimum value of $f(x)$.

5. Show that $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x + 8$ has maximum value at $x = 2$ and minimum at $x = -1.5$.

6. Find the maximum and minimum value of the function $f(x) = x^3 - 3x^2 - 9x + 5$.

7. Find the maximum and minimum value of the function

$$f(x) = \frac{x^3}{3} + \frac{3}{2}x^2 - 28x + 75.$$

8. Show that $f(x) = x^3 - 9x^2 + 30x + 5$ has neither a maximum nor a minimum.

9. Find two positive numbers whose sum is 40 and product is maximum.

10. The demand curve and supply curve of a commodity are given by $D: x = 19 - 3p - p^2$ and $S: x = 2p - 1$. Find the equilibrium price and quantity exchanged.

11. The demand and supply function of a commodity are as follow :

$D : (x + 1)(p + 2) = 60;$ $\quad$ $S : x = p - 6.$

Find equilibrium price and quantity.

12. Find the equilibrium price and quantity using the given demand and supply function.

Demand : $x = 16 - 8p$; Supply ; $x = 32 + 8p$.

13. If the demand function $x = 120$. Then find out elasticity of demand at $p = 2$.

14. When the price of brand of soap was Rs. 12 per piece, its demand was 1000 pieces. When the price of this soap increased to Rs. 16, its demand decreased to 800 pieces. Find the elasticity of demand for the soap.

15. The demand function of the monopolist is $x = 25 - 4p + p^2$. Calculate the elasticity of demand at $p = 6$.

16. A fertilizer company produces x tons of fertilizer per month at the cost of Rs. $C = \frac{x^3}{3} - 3x^2 - 16x$. What should be the production for minimum expenditure?

17. If $C = x^3 - 315x^2 + 27000x + 20000$ is a total production cost function where x denotes number of units produced. How many units should produce to minimize total cost.

18. If cost function $C(x) = \frac{x^2}{35} - 4x + 840$. Find production x for minimum cost and also find minimum cost.

19. If cost function $C = x^2 + 4x + 4$ then find,

(1) Average cost function (2) Marginal cost function

(3) Output (x) for which $MC = AC$.

20. If the demand function of a commodity is $p = 120 - 4x^2$. Find the revenue function and determine the maximum revenue. What should be the price and demand for getting maximum revenue ?

21. The demand function of a commodity is $p = 15 - \frac{x^2}{50}$. Find its total revenue function. Also find the Marginal revenue when demand is 5 unit.

22. If the demand function $p = 12 - 4x$ and cost function $C = 8x - x^2$ of a commodity. Find maximum profit. What will be the price and demand to get maximum profit.

23. A producer is manufacturing x radio sets per day. The demand function of radio set $x = 100 - 2\sqrt{p}$. The cost function of x sets per day $C = \frac{x^2}{30} + 3x + 50$. Find the profit function. How many radio set should be manufactured per day to get maximum profit. Also determine the price for maximum profit.

24. Production cost function of a company is $C = 500 + x^2$ and revenue function is R = $200x$. How many units should be produced in order to maximise the profit ? Find maximum profit.

25. If the consumption function is given by $C = 16 + 9\, I^{\frac{3}{2}}$, determine the marginal propensity to consume and marginal propensity to save when $I = 25$.

:: ANS ::

2. *At* $x = 2$, Maxi.= 22 and at $x = 4$, Mini.= 18

3. At $x = 1$, Maxi.= 9 and at $x = 3$, Mini.= 5

4. At $x = 2 - \sqrt{7}$ function is maxi.= 22 and at $x = 2 + \sqrt{7}$ function is Minimum.

6. At $x = -1$, Maxi.= 10 and at $x = 3$, Mini.= -22

7. At $x = 4$, Maxi.= $\frac{569}{3}$ and at $x = -7$, Mini.= $\frac{401}{6}$

9. ($x = 20$, $y = 20$) **10.** $p = \frac{\sqrt{105} - 5}{2}$, $x = \sqrt{105} - 6$ **11.** $p = 10$, $x = 4$

12. $p = \frac{1}{2}$, $x = 12$ **13.** $\eta_d = -\frac{2}{239}$ **14.** $\eta_d = \frac{7}{9}$

15. $\eta_d = -\frac{48}{37}$ **16.** $x = 8$ **17.** $x = 150$

18. $x = 70$, *Minimum Cost* = 340.

19. (1) $AC = x + 4 + \frac{4}{x}$ **(2)** MC = $2x + 4$ **(3)** $x = 2$

20. *Maximum Revenue* = 253, $p = 80$, $x = \sqrt{10}$

21. $x = \frac{27}{2}$ **22.** $x = \frac{2}{3}$, *Maximum profit* $= P = \frac{4}{3}$, $p = \frac{28}{3}$

23. $x = 27$ **24.** $x = 200$, *Maximum Profit* = 19500

25. MPS = 67.5, MPS = –66.5

➢ **Select the appropriate answer from the given alternative answer. (M.C.Q.)**

1. The maximum value of the function $f(x) = 2x^3 + 9x^2 - 60x + 5$ is

(a) 2 (b) –5 (c) –63 (d) 180

2. The minimum value of the function $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 6x + 8$ is

(a) 1.5 (b) 0.125 (c) -2 (d) none of these

3. If the demand and supply functions are respectively $D = 55 - 2p$ and $S = 20 + 1.5p$ then equilibrium is

(a) 35 (b) 10 (c) 18 (d) 22

4. The demand functions for a commodity is $p = 10 + 5x - 2x^2$. Then elasticity of demand at $x = 5$ is

(a) 5 (b) 10 (c) -0.5 (d) -0.2

5. Cost function of manufacturing certain product is Rs. $\frac{x^3}{3} - 5x^2 - 200x$. Then marginal cost function is

(a) $2x - 10$ (b) –10 (c) $x^2 - 10x - 200$ (d) –5

6. Demand function of manufacturing certain commodity is Rs. $p = 30 - \frac{x}{2}$. Then marginal revenue function is

(a) $30 + x$ (b) 30 (c) $30x - x^2$ (d) $30 - x$

7. The demand and cost function of a commodity for a monopolist are

$p = 40 - x$ and $C = 10 + 5x + \frac{1}{4}x^2$. The profit function for production is

(a) $40x + x^2$ (b) $35x - \frac{5}{4}x^2 - 10$ (c) $30x - x^2$ (d) none of these

8. The profit function of producing x items is $P = 2.7x - 50 - 0.001x^2$. Then the production rate x that will maximises profit is.

(a) 250 (b) 2.7 (c) 350 (d) none of these

9. $\frac{d^2y}{dx^2}$ of $y = 5x^4 - 20x + 37$ is

(a) $60x^2$ (b) $60x^2 - 20x$ (c) $60x^2 - 20$ (d) none of these

10. The demand and supply function of a commodity are as follow :

$D : (x + 10)(p + 20) = 300$ $S : x = 2p - 8$.

Then equilibrium price is.

(a) 3 (b) 4 (c) 5 (d) 6

:: ANS ::

1.	(d)	**2.**	(b)	**3.**	(a)	**4.**	(d)	**5.**	(c)
6.	(d)	**7.**	(b)	**8.**	(c)	**9.**	(a)	**10.**	(c)

UNIT-11 INTEGRATION

11.1 Introduction to Integration

What is Integration?

Integration is a fundamental concept in calculus that represents the process of finding the whole from its parts. In mathematical terms, integration is the inverse operation of differentiation. While differentiation breaks down a function to understand its rate of change, integration combines these rates of change to reconstruct the original function or to find the accumulated value over an interval.

Why is Integration Important?

Integration is crucial in many fields, including business, economics, physics, and engineering. It allows us to:

1. Calculate areas under curves
2. Find volumes of solids
3. Compute total values from rates of change
4. Analyze continuous probability distributions
5. Solve differential equations

In business contexts, integration helps in understanding total costs, revenues, and profits over time, analyzing consumer and producer surplus, and evaluating investments with continuous cash flows.

11.2 Definition and Basic Concepts

How is Integration defined mathematically?

Integration is the process of finding a function F(x) whose derivative is the given function f(x). It is also called antiderivative of a function.The mathematical notation for integration is:

$$\int f(x)dx = F(x) + C$$

Where:

- $\int$ is the integral symbol, derived from the Latin word "summa"
- f(x) is the integrand, the function being integrated
- dx indicates that x is the variable of integration

Types of Integrals:

1. **Indefinite Integrals**
   An indefinite integral represents a family of antiderivatives and is expressed as:

   $$\int f(x)dx = F(x) + C$$

   where F(x) is the antiderivative of f(x) and C is the constant of integration. The constant of integration is necessary because the derivative of a constant is zero, so there are infinitely many functions whose derivative is f(x), differing only by a constant.

2. **Definite Integrals**
   A definite integral calculates the signed area between the curve of a function and the x-axis over a specific interval [a,b]:

   $$\int_a^b f\,(x)dx = F(b) - F(b) + C$$

   This formula is known as the Fundamental Theorem of Calculus, which establishes the connection between differentiation and integration.

How is Integration Related to Differentiation?

Integration and differentiation are inverse operations. If F(x) is an antiderivative of f(x), then:

$$\frac{d}{dx}\,[F(x)] = f(x)\ and \int f(x)dx = F(x) + C$$

This relationship is the basis of the Fundamental Theorem of Calculus and is crucial for solving many integration problems.

11.3 Basic Rules of Integration

How do we integrate common functions?
Several basic rules form the foundation for integrating various functions:

1. **Constant Rule:**

$$\int k dx = kx + C$$

   This rule states that the integral of a constant is the constant multiplied by the variable, plus the constant of integration.

2. **Power Rule:**

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

   , where $n \neq -1$

3. **Sum Rule:**

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

   This rule allows us to integrate functions term by term.

4. **Difference Rule:**

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

   Similar to the sum rule, we can separate the integral of a difference.

5. **Constant Multiple Rule:**

$$\int k \cdot f(x)dx = k\int f(x)dx, \; where\ k\ is\ a\ constant.$$

   Constants can be factored out of an integral.

6. **Exponential Rule:**

$$\int e^x dx = e^x + C$$

7. **Trigonometric Rules:**

   - $\int sin(x)\, dx = -cos(x) + C$
   - $\int cos(x)\, dx = sin(x) + C$
   - $\int tan(x)\, dx = -ln|cos(x)| + C$

   These basic rules form the foundation for more complex integration techniques.

11.4 Techniques of Integration

How do we integrate more complex functions?

In calculus, integrating complex functions often requires techniques beyond basic antiderivatives. As functions become more intricate, direct integration methods may not apply, and advanced strategies are necessary to simplify the process. Techniques like substitution, integration by parts, and the integration of rational functions allow us to decompose and manipulate complex integrals, making them more approachable. Each method provides a systematic approach to break down functions into simpler components, enabling us to solve integrals that would otherwise be challenging or impossible with elementary techniques. These tools become essential in tackling integrals that arise in real-world applications, such as physics, engineering, and economics.

11.4.1 Integration by Substitution

What is Integration by Substitution?

Integration by substitution, also known as u-substitution, is a method used to simplify complex integrals by introducing a new variable of integration.
How Does It Work??

1. Choose a substitution $u = g(x)$
2. Calculate $\frac{du}{dx}$
3. Express the integral in terms of u
4. Integrate with respect to u
5. Substitute back to express the result in terms of x

Example 1: Integrate $\int x(x^2 + 1)^3\, dx$
Let $u = x^2 + 1$
Then $du = 2x\, dx$, or $x\, dx = 1/2\, du$
Rewriting the integral:

$$\int x(x^2 + 1)^3\, dx = \int 1/2u^3\, du = 1/8u^4 + C = 1/8(x^2 + 1)^4 + C$$

Example 2: Integrate $\int \frac{dx}{(p+qx)^4}$
Let I = $\int \frac{dx}{(p+qx)^4}$
Assume $p + qx = r \Rightarrow qdx = dr \Rightarrow dx = \frac{1}{q}dr$
Rewriting the integral:

$$I = \int \frac{dx}{(p + qx)^4} = \frac{1}{q}\int r^{-4}\, dr = \frac{1}{q}\frac{r^{-3}}{(-3)} + C$$

$$I = \frac{-1}{3q}(p + qx)^{-3} + C = \frac{-1}{3q(p + qx)^3} + C$$

Example 3: Integrate $\int sin^2\ x\ dx$
Let

$$\begin{aligned} I &= \int sin^2\ x\ dx \\ &= \frac{1}{2}\int (1 - cos2x)\ dx \\ &= \frac{1}{2}\int\ dx - \frac{1}{2}\int cos2x\ dx \\ &= \frac{1}{2}\ x - \frac{1}{2}\ \frac{sin2x}{2} + C \\ &= \frac{1}{2}\ x - \frac{1}{4}\ sin2x + C \end{aligned}$$

11.4.2 Integration by Parts

What is Integration by Parts?

Integration by parts is a technique based on the product rule of differentiation. It is particularly useful for integrals involving products of functions.

How is it formulated?

The formula for integration by parts is:

$$\int u\ dv = uv\ - \int v\ du$$

Where u and dv are chosen to simplify the integral.

Example 1: Integrate $\int x\ ln(x)\ dx$
Let $u = ln(x), dv = x\ dx$
Then $du = 1/x\ dx, v = x^2/2$

$$\int x\ ln(x)\ dx = (x^2/2)\ ln(x) - \int (x^2/2)\ (1/x)\ dx = (x^2/2)\ ln(x) - x^2/4 + C$$

Example 2: Integrate $\int x\ cosx\ dx$

$$\begin{aligned} I &= \int x\ cosx\ dx \\ &= x\int cosx\ dx - \int\ \left[\frac{d}{dx}(x)\ .\ \int cosx\ dx\right] dx \\ &= x\ sinx - \int 1. sinx. dx \\ &= x\ sinx + cosx + C \end{aligned}$$

Example 3: Integrate $\int x^2\ sinx\ dx$

$$\begin{aligned} I &= \int x^2\ sinx\ dx \\ &= x^2\int sinx\ dx - \int\ \left[\frac{d}{dx}(x^2)\ .\ \int sinx\ dx\right] dx \\ &= x^2\ (-cosx) - \int 2x.(-cosx). dx \\ &= -x^2\ cosx + 2\int xcosx\ dx \\ &= -x^2\ cosx + 2\left[x\ \int cosx\ dx - \int \left\{1\int cosx\ dx\ \right\} dx\right] \\ &= -x^2\ cosx + 2\left[x\text{sin}x - \int sinx\ dx\right] \\ &= -x^2\ cosx + 2[x\text{sin}x + cosx] + C \end{aligned}$$

11.4.3 Integration of Rational Functions

What are rational functions?

Rational functions are functions that can be expressed as the ratio of two polynomials.

How do we integrate them?

The key technique for integrating rational functions is partial fraction decomposition. This method involves breaking down complex fractions into simpler ones that can be integrated more easily. The steps are as follows:

1. Ensure the denominator's degree is greater than the numerator's. If not, perform long division first.
2. Factor the denominator.
3. Set up partial fractions based on the factors of the denominator.
4. Solve for the coefficients of the partial fractions.
5. Integrate each partial fraction separately.

Example 1: Integrate $\int (2x+1)/(x^2-1)\ dx$

Partial fraction decomposition $\frac{2x+1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

Solving for A and B, we get $A = 3/2$ and $B = -1/2$

So,

$$\int \frac{(2x+1)}{(x^2-1)}\ dx = \frac{3}{2}\int \frac{1}{(x-1)}\ dx - \frac{1}{2}\int \frac{1}{x+1}\ dx = \frac{3}{2} ln|x-1| - (1/2) ln|x+1| + C$$

Example 2: Integrate $\int \frac{1}{2x^2+x-1} dx$

$$\begin{aligned} 2x^2+x-1 &= 2x^2+2x-x-1 \\ &= 2x(x+1) - 1(x+1) \\ &= (2x-1)(x+1) \end{aligned}$$

Partial fraction decomposition:

$$\begin{aligned} \frac{1}{2x^2+x-1} &= \frac{A}{x+1} + \frac{B}{2x-1} \\ &= \frac{A(2x-1)+B(x+1)}{(2x-1)(x+1)} \end{aligned}$$

$$\therefore\ A(2x-1) + B(x+1) = 1$$

Taking $x = \frac{1}{2}$ and $x = -1$, we get $A = \frac{-1}{3}$ and $B = \frac{2}{3}$

$$\therefore\ \frac{1}{2x^2+x-1} = \frac{-1}{3(x+1)} + \frac{2}{3(2x-1)}$$

$$\therefore \int \frac{1}{2x^2+x-1}dx = \frac{-1}{3}\int \frac{dx}{x+1} + \frac{2}{3}\int \frac{dx}{2x-1}$$
$$= \frac{-1}{3}\ log(x+1) + \frac{2}{3}.\frac{1}{2}\ log(2x-1) + C$$
$$= \frac{1}{3}\ log(2x-1) - \frac{1}{3}\ log(x+1) + C$$
$$= \frac{1}{3}\ [log(2x-1) - log(x+1)] + C$$
$$\int \frac{1}{2x^2+x-1}dx = \frac{1}{3}\ log\frac{2x-1}{x+1} + C$$

11.5 Numerical Methods for Integration

Why Use Numerical Methods?

Numerical methods are used when analytical solutions are difficult or impossible to obtain. They provide approximate solutions to definite integrals.

11.5.1 Trapezoidal Rule

What is the Trapezoidal Rule?

The trapezoidal rule approximates the area under a curve by dividing it into trapezoids. **How is it formulated?**

For n subintervals:

$$\int_a^b f(x)\ dx \approx \frac{(b-a)}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

where $x_0 = a$, $x_n = b$, and x_i are equally spaced points between a and b.

Example 1:

Approximate the integral $\int_0^1 x^2\ \ dx$ using the trapezoidal rule with four subintervals.

The endpoints of the subintervals as the set $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $\Delta x = \frac{1-0}{4}$. Thus,

$$\int_0^1 x^2\ \ dx \approxeq \frac{1}{2}\frac{1}{4}\left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right)$$
$$= \frac{1}{8}\left(0 + 2.\frac{1}{16} + 2.\frac{1}{4} + 2.\frac{9}{16} + 1\right)$$
$$= \frac{11}{32}$$

Example 2:

Approximate the integral $\int_2^4 x^3\ \ dx$ using the trapezoidal rule with four subintervals.

The endpoints of the subintervals as the set $P = \left\{2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$ and $\Delta x = \frac{4-2}{4}$. Thus,

$$\begin{aligned}\int_2^4 x^3\ dx &\approxeq \frac{1}{2}\frac{1}{2}\left(f(2)+2f\left(\frac{5}{2}\right)+2f(3)+2f\left(\frac{7}{2}\right)+f(4)\right)\\ &=\frac{1}{4}\left(2^3+2.\left(\frac{5}{2}\right)^3+2.(3)^3+2.\left(\frac{7}{2}\right)^3+4^3\right)\\ &=\frac{243}{4}\end{aligned}$$

11.5.2 Simpson's Rule

What is Simpson's Rule?

Simpson's rule uses parabolic arcs to approximate the area under a curve, often providing more accurate results than the trapezoidal rule.

How is it formulated?

For n even subintervals:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

where $x_0 = a$, $x_n = b$, and x_i are equally spaced points between a and b.

Example 1:

Approximate the integral $\int_0^3 x^2\ dx$ using the Simpson's rule with four subintervals.

The endpoints of the subintervals as the set $P = \left\{0, \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3\right\}$ and $\Delta x = \frac{3-0}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2\ dx &\approxeq \frac{1}{3}\frac{3}{4}\left(f(0)+4f\left(\frac{3}{4}\right)+2f\left(\frac{3}{2}\right)+4f\left(\frac{9}{4}\right)+f(3)\right)\\ &=\frac{1}{4}\left(0+4.\frac{9}{16}+2.\frac{9}{4}+4.\frac{81}{16}+9\right)\\ &=9\end{aligned}$$

Example 2:

Approximate the integral $\int_2^4 x^3\ dx$ using the Simpson's rule with four subintervals.

The endpoints of the subintervals as the set $P = \left\{2, \frac{5}{2}, 3, \frac{7}{2}, 4\right\}$ and $\Delta x = \frac{4-2}{4}$. Thus,

$$\begin{aligned}\int_2^4 x^3\ dx &\approxeq \frac{1}{3}\frac{1}{2}\left(f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+f(4)\right)\\ &=\frac{1}{6}\left(2^3+4.\left(\frac{5}{2}\right)^3+2.(3)^3+4.\left(\frac{7}{2}\right)^3+4^3\right)\\ &=60\end{aligned}$$

These numerical methods are particularly useful in business applications where exact analytical solutions may not be feasible or necessary.

11.6 Illustration

Find the indefinite integral of $f(x) = x$.

Solution

$$\int x\,dx = \frac{x^2}{2} + C$$

Here are some examples to provide better insights into the relationship between derivatives and indefinite integrals.

1. $\frac{d}{dx}\ logx = \frac{1}{x} \Rightarrow \int \frac{1}{x}\ dx = logx + C$
2. $\frac{d}{dx}\ e^x = e^x \Rightarrow \int e^x\ dx = e^x + C$
3. $\frac{d}{dx}\ x^2 = 2x \Rightarrow \int 2x\ dx = x^2 + C$

$$\int x^4\ dx = \frac{x^{4+1}}{4+1} = \frac{1}{5}\ x^5 + C$$

$$\int \sqrt{x}\ dx = \int x^{\frac{1}{2}}\ dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}$$

4. Evaluate the following: $\int \frac{(1+x)^3}{x^2}\ dx$

$$\begin{aligned} Let\ I\ &= \int \frac{(1+x)^3}{x^2}\ dx \\ &= \int \frac{1+3x+3x^2+x^3}{x^2}\ dx \\ &= \int \frac{1}{x^2}\ dx + \int \frac{3}{x}\ dx + \int 3\ dx + \int x\ dx \\ &= \int x^{-2} dx + 3\int \frac{1}{x}\ dx + 3\int 1\ dx + \int x\ dx \\ &= \frac{x^{-1}}{-1} + 3\ logx + 3x + \frac{x^2}{2} + C \end{aligned}$$

5. Evaluate the following: $\int \frac{x^4}{x^2+1}\ dx$

$$\begin{aligned} Let\ I\ &= \int \frac{x^4}{x^2+1}\ dx \\ &= \int \frac{(x^4-1)+1}{x^2+1}\ dx \\ &= \int \frac{(x^2-1)(x^2+1)+1}{x^2+1}\ dx \\ &= \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx \\ &= \int x^2\ dx - 1\int dx + \int \frac{1}{x^2+1}\ dx \\ &= \frac{x^3}{3} - 1.x + tan^{-1}(x) + C \end{aligned}$$

6. Integrate the following: $\int (5x-2)^3\ dx$

Let I = $\int (5x-2)^3\ dx$

Substitute $5x-2=u \Rightarrow 5dx=du \Rightarrow dx=\frac{1}{5}.du$

$$\begin{aligned} I &= \int (5x-2)^3\ dx \\ &= \int u^3.\frac{1}{5}.du \\ &= \frac{1}{5}\int u^3\ du \\ &= \frac{1}{5}\frac{u^4}{4}+C \\ &= \frac{1}{20}u^4+C \\ I &= \frac{1}{20}(5x-x)^4+C \end{aligned}$$

7. Integrate the following: $\int x.cos^2\ x\ dx$

$$\begin{aligned} Assume\ I &= \int x.cos^2\ x\ dx \\ I &= \frac{1}{2}\int x(2cos^2\ x)\ dx = \frac{1}{2}\int x(1+cos2x)dx \\ &= \frac{1}{2}\left[\int x\ dx + \int x\ cos2x\ dx\right] \\ &= \frac{1}{2}\left[\frac{x^2}{2}+x\int cos2x\ dx - \int\left\{1.\int cos2x\ dx\right\}dx\right] \\ &= \frac{1}{2}\left[\frac{x^2}{2}+\frac{x\ sin2x}{2}-\int\ \frac{sin2x}{2}\ dx\right] \\ &= \frac{1}{2}\left[\frac{x^2}{2}+\frac{x\ sin2x}{2}+\frac{1}{2}\frac{cos2x}{2}\ dx\right] \\ &= \frac{1}{4}\left[x^2+x\ sin2x+\frac{cos2x}{2}\right]+C \end{aligned}$$

8. Integrate the following: $\int x.sec^2\ (x)\ dx$

$$\begin{aligned} Assume\ I &= \int x.sec^2\ (x)\ dx \\ I &= x\int sec^2\ (x)\ dx - \int 1.\left(\int sec^2\ (x)\ dx\right)dx \\ &= x\ tan(x) - \int tan(x)\ dx \\ &= x\ tan(x) + log\ cos(x) + C \end{aligned}$$

9. Integrate the following: $\int \frac{log(x)}{x^2}\ dx$

$$\begin{aligned}
Assume\ I &= \int \frac{log(x)}{x^2}\ dx \\
&= \int log(x).\ \frac{1}{x^2}.\ dx \\
&= log(x)\int \frac{1}{x^2}\ dx - \int \left[\frac{1}{x} \int \frac{1}{x^2}\ dx\right] dx \\
&= log(x)\left(\frac{-1}{x}\right) - \int \frac{1}{x}\left(\frac{-1}{x}\right) dx \\
&= \frac{-1}{x}.\ log(x) + \int x^{-2}\ dx \\
&= \frac{-1}{x}.\ log(x) - \frac{1}{x} + C \\
&= \frac{-1}{x}(1 + logx) + C
\end{aligned}$$

10. Integrate the following: $\int \frac{3x}{(x-1)(x-2)(x-3)}\ dx$

Let $\frac{3x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$

$\Rightarrow 3x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$

$Replacing\ x\ by\ 1,\ 2\ \&\ 3\ on\ both\ sides\ we\ get$

$3 \times 1 = A(1-2)(1-3) \Rightarrow A = \frac{3}{2}$

$3 \times 2 = B(2-1)(2-3) \Rightarrow B = -6$

$3 \times 3 = C(3-1)(3-2) \Rightarrow C = \frac{9}{2}$

$$\begin{aligned}
\frac{3x}{(x-1)(x-2)(x-3)} &= \frac{3}{2(x-1)} - \frac{6}{x-2} + \frac{9}{2(x-3)} \\
\int \frac{3x}{(x-1)(x-2)(x-3)}\ dx &= \frac{3}{2}\int \frac{dx}{x-1} - 6\int \frac{dx}{x-2} + \frac{9}{2}\int \frac{dx}{x-3} \\
&= \frac{3}{2}\ log(x-1) - 6\ log(x-2) + \frac{9}{2}\ log(x-3) + C
\end{aligned}$$

11.7 Exercises and Problems

1. $\int (3x^2 + 2x - 5)\ dx$

2. $\int (1/x)\ dx$

3. $\int e^x dx$

4. $\int \sin(x)\ dx$

5. Find the definite integral:

$$\int_0^{\frac{\pi}{2}} c\ os(x)\ dx$$

6. Solve using Integration by Substitution $\int x(x^2+1)^4 dx$

7. Solve using Integration by Parts $\int x\ cos(2x) dx$

8. Evaluate the following integral using partial fraction decomposition:

$$\int \frac{(x+1)}{(x^2-1)}\ dx$$

9. Evaluate the following integral using partial fraction decomposition:

$$\int \frac{(2x+3)}{(x^2+2x+1)}\ dx$$

10. Use the trapezoidal rule with 4 subintervals to approximate: $\int_0^1 x^2\ \ dx$

11. Use Simpson's rule with 4 subintervals to approximate: $\int_0^\pi \sin(x)\ dx$

12. Which of the following is the integral of 2x?

a. x^2

b. $x^2 + C$

c. $x^2 + 2$

d. $2x^2 + C$

13. The integral $\int_a^b f(x)\ dx$represents:

a. The slope of f(x) at x = a
b. The area under the curve f(x) from x = a to x = b
c. The value of f(x) at x = b
d. The difference between f(a) and f(b)

14. Short Answer Questions

a. Explain the relationship between differentiation and integration.
b. Describe the difference between definite and indefinite integrals.
c. When would you use numerical integration methods instead of analytical methods?

UNIT-12 APPLICATIONS OF INTEGRATION

12.1 Definite Integration

If $f(x)$ is continuous on [a, b] and if F(x) is any indefinite integral of f(x); then

$$\int_a^b f(x).\ dx = F(b) - F(a)$$

and the integral $\int_a^b f(x).\ dx$ is called definite integral within limits a and b. Note that a is the upper limit and b is the lower limit for this integral. Have a look at the following example & evaluate:

1. $\int_1^2 x^2\ dx$

$$Let\ I_1 \quad = \int_1^2 x^2\ dx = \left[\frac{x^3}{3}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3}$$

2. $\int_1^3 \frac{dx}{x}$

$$Let\ I_2 \quad = \int_1^3 \frac{dx}{x} = [log\ x]_1^3 = [log\ 3 - log\ 1] = log3$$

3. $\int_0^1 \frac{1}{2x-3}\, dx$

$$Let\ I_3 = \int_0^1 \frac{1}{2x-3}\, dx$$
$$= \left[\frac{1}{2}\ log\ (2x-3)\right]_0^1$$
$$= \frac{1}{2}[log\ (2x-3)]_0^1$$
$$= \frac{1}{2}[log\ |(-1)| - log\ |(-3)|]$$
$$= \frac{1}{2}[log\ |1| - log\ |3|]$$
$$= \frac{-1}{2} log\ 3$$

12.2 Introduction to Business Applications of Integration

Why is Integration Important in Business?

Integration plays a crucial role in various business and economic applications. It allows us to:

1. Calculate total quantities from rates of change.
2. Determine areas under curves, which often represent economic concepts.
3. Compute volumes, useful in inventory and production management.
4. Analyze continuous cash flows for financial planning.
5. Evaluate economic welfare and market efficiency.

How Does Integration Connect to Business Concepts?

Many business phenomena occur continuously over time or across a range of values. Integration provides a tool to aggregate these continuous effects, enabling more accurate analysis and decision-making in areas such as finance, economics, and operations management.

12.3 Area Between Curves

What Does the Area Between Curves Represent in Economics?

In economics, the area between two curves often represents an economic surplus or the difference between two cumulative quantities.

12.3.1 Consumer and Producer Surplus

What is Consumer Surplus?

Consumer surplus is the difference between the maximum price a consumer is willing to pay for a good or service and the actual price they pay. Graphically, it's represented by the area between the demand curve and the price line. **What is Producer Surplus?** Producer surplus is the difference between the price producers receive for their goods and the minimum price they would be willing to accept. Graphically, it's the area between the price line and the supply curve.

How Do We Calculate These Surpluses Using Integration?

To calculate consumer surplus:

$$CS \ = \int_{P}^{P_{max}} D\,(P)dP - P * (Q_d)$$

where:

- D(P) is the demand function
- P is the market price
- P_{max} is the price at which quantity demanded becomes zero
- Q_d is the quantity demanded at price P

For producer surplus:

$$PS \ = P * (Q_s) - \int_{0}^{P} S\,(P)dP$$

where:

- S(P) is the supply function
- Q_s is the quantity supplied at price P

1. Calculate the consumer surplus if the demand function is $P = 100 - Q$ and the market price is 60.

$$\begin{aligned} Consumer\ Surplus \quad &= \int_{60}^{100} (100 - Q)\,dQ - 60 * (100 - 60) \\ &= \left[100Q - \frac{Q^2}{2}\right]_{60}^{100} - \ 2400 \\ &= (10000 - 5000) - (6000 - 1800) - 2400 \\ &= 800 \end{aligned}$$

Why is this important?

Understanding consumer and producer surplus helps in analyzing market efficiency, the impact of policies, and overall economic welfare.

12.4 Present and Future Value of Continuous Cash Flows

How Does Integration Apply to Financial Calculations?

In finance, many cash flows occur continuously rather than at discrete intervals. Integration allows us to calculate the present or future value of these continuous cash flows.

What is the Formula for Continuous Present Value?

The present value of a continuous cash flow from time 0 to time T is given by:

$$PV = \int_0^T C(t) * e^{-rt}\, dt$$

where:

- C(t) is the cash flow function
- r is the continuous interest rate
- t is the time function

What is the Formula for Continuous Future Value?

The future value formula is similar to the present value, but without the discounting factor:

$$FV = \int_0^T C(t) * e^{-r(T-t)} dt$$

1. Calculate the present value of a continuous cash flow of $1000 per year for 5 years, with a continuous interest rate of 5%.

$$\begin{aligned} PV &= \int_0^5 1000 * e^{-0.05t}\, dt \\ &= -20000 * [e^{-0.05t}]_0^5 \\ &= -20000 * [e^{-0.25} - 1] \\ &\approx 4877.31 \end{aligned}$$

Why is This Important in Business?

These calculations are crucial for investment analysis, project valuation, and long-term financial planning, especially when dealing with continuous revenue or cost streams.

12.5 Break-even Analysis

How can integration be used in Break-Even Analysis?

In cases where revenue and cost functions are continuous, integration can be used to determine the break-even point.

What's the Process?

Define continuous functions for revenue R(q) and cost C(q).

Find the break-even quantity q* where R(q*) = C(q*). Use integration to calculate total revenue and total cost up to q*.

12.5.1 Example

1. Suppose R(q) = $100q - 0.1q^2$ and C(q) = $2000 + 20q + 0.05q^2$. Find the break-even quantity and the total revenue and cost at this point.
   Solving $100q - 0.1q^2 = 2000 + 20q + 0.05q^2$, we get $q^* \approx 54.77$units
   Total Revenue = $\int_0^{54.77} 100\,q - 0.1q^2\ dq \approx 4095.37$
   Total Cost =$\int_0^{54.77}(20q + 0.05q^2)\ dq + 2000 \approx 4095.37$

Why is this Important?

This analysis helps businesses understand at what point they start making a profit, which is crucial for pricing strategies and production planning.

12.6 Marginal Analysis in Economics

How does integration relate to Marginal Analysis?

Integration allows us to move from marginal (rate of change) functions to total functions, which is crucial in economic analysis.

12.6.1 Total Cost from Marginal Cost

What's the relationship between Marginal and Total Cost?

The total cost function can be derived by integrating the marginal cost function and adding fixed costs:

$$TC\ = \int MC(q)dq + FC$$

where:

- TC is Total Cost
- MC is Marginal Cost
- FC is Fixed Cost

1. If MC(q) = 10 + 0.2q and fixed costs are $1000, find the total cost function.

   TC = $\int (10 + 0.2q)\, dq + 1000$

   $=10q + 0.1q^2 + 1000 + C$

12.6.2 Total Revenue from Marginal Revenue

How Do We Find Total Revenue from Marginal Revenue?

Similarly, the total revenue function can be found by integrating the marginal revenue function:

$$TR = \int MR(q)\, dq$$

Why is This Important?

These relationships allow businesses to understand how their costs and revenues change as production quantities change, which is crucial for profit maximization and strategic decision-making.

12.6.3 Income Distribution: Lorenz Curves and Gini Coefficient

What is a Lorenz Curve?

A Lorenz curve is a graphical representation of income distribution in a population. It plots the cumulative percentage of income against the cumulative percentage of the population.

How is the Gini Coefficient Calculated?

The Gini coefficient, a measure of income inequality, is calculated using the area between the Lorenz curve and the line of perfect equality:

$$Gini\ Coefficient = \frac{Area\ between\ Lorenz\ curve\ and\ line\ of\ equality}{Total\ area\ under\ line\ of\ equality}$$

This can be calculated using integration:

$$Gini\ Index = 1 - 2 * \int_0^1 L\,(x)\, dx$$

Where L(x) is the Lorenz curve function.

Why is This Important in Business and Economics?

Understanding income distribution is crucial for businesses in market segmentation, pricing strategies, and for policymakers in addressing economic inequality.

12.7 Inventory Management

How is integration used in Inventory Management?

Integration can be used to calculate total holding costs and ordering costs over time, helping to determine optimal order quantities.

What's the Process?

Define functions for holding cost rate h(t) and demand rate D(t).

Integrate these functions over the inventory cycle to get total costs. Optimize to find the best order quantity.

1. If holding cost is \$2 per unit per year and demand is constant at 1000 units per year, find the total holding cost for a cycle of Q units.

   Total Holding Cost = $\int_0^T 2 * (Q - 1000t)\ dt$

   where T=Q/1000 (time to use up Q units)

   This gives: Total holding cost = $Q^2/1000$

Why is This Important?

This analysis helps businesses minimize their inventory costs while ensuring they can meet customer demand.

12.8 Illustrations

Example 1: The demand function for a product is $P = 200 - 2Q$, and the supply function is $P = 50 + 3Q$, where P is price and Q is quantity. Find the equilibrium price and quantity. Also calculate the consumer and producer surplus.

At equilibrium: $200 - 2Q = 50 + 3Q \Rightarrow 150 = 5Q$

$$\therefore\ Q = 30, P = 140$$

Consumer Surplus

$$\begin{aligned} CS &= \int_{30}^{100} (200 - 2Q)\ dQ - 140(30) \\ &= [200Q - Q^2]_{30}^{100} - 4200 \\ &= (20000 - 10000) - (6000 - 900) - 4200 \\ &= 700 \end{aligned}$$

Producer Surplus

$$\begin{aligned} PS &= 140(30) - \int_{30}^{100} (50 + 3Q)\ dQ \\ &= 4200 - [50Q + 1.5Q^2]_0^{30} \\ &= 4200 - (1500 + 1350) \\ &= 1350 \end{aligned}$$

Example 2: In a labor market, the supply of labor (in hours) is given by $S = 2W - 10$, and the demand for labor is given by $D = 50 - W$, where W is the hourly wage. Calculate the worker and employer surplus at equilibrium.

At Equilibrium: $2W - 10 = 50 - W \Rightarrow 3W = 60 \Rightarrow W = 20, Q = 30$
Worker Surplus

$$\begin{aligned} WS &= \int_5^{20} \left(20 - \frac{Q+10}{2}\right) dQ \\ &= [20Q - 0.25Q^2 - 5Q]_5^{20} \\ &= (400 - 100 - 100) - (100 - 6.25 - 25) \\ &= 131.25 \end{aligned}$$

Employer Surplus

$$\begin{aligned} ES &= \int_{20}^{50} (50 - Q)\, dQ - 20(30) \\ &= [50Q - 0.5Q^2]_{20}^{50} - 600 \\ &= (2500 - 1250) - (1000 - 200) - 600 \\ &= 225 \end{aligned}$$

Example 3: A company expects its costs to grow continuously according to the function $C(t) = 50000e^{(0.03t)}$, where t is in years. Calculate the present value of the cost stream for the next 8 years if the discount rate is 5%

For calculation of Present Value

$$\begin{aligned} PV &= \int_0^8 50000\, e^{(0.03t)} e^{(-0.05t)}\, dt \\ &= 50000 \int_0^8 e^{(-0.02t)}\, dt \\ &= \left[-2500000e^{(-0.02t)}\right]_0^8 \\ &= -2500000\left(e^{(-0.16)} - 1\right) \\ &\approx 372{,}173.07 \end{aligned}$$

Example 4: An oil well's production is expected to decline according to the function $Q(t) = 10000e^{(-0.1t)}$ barrels per year, where t is in years. If the price of oil is expected to remain constant at $50 per barrel and the discount rate is 8%, what is the present value of the revenue stream over the next 15 years?

For calculation of Present Value

$$
\begin{aligned}
PV &= \int_0^{15} 50\left(10000e^{(-0.1t)}\right)e^{(-0.08t)}\,dt \\
&= 50000\int_0^{15} e^{(-0.18t)}\,dt \\
&= \left[--2777777.78e^{(-0.18t)}\right]_0^{15} \\
&= -2777777.78\left(e^{(-2.7)}-1\right) \\
&\approx 2{,}403{,}273.90
\end{aligned}
$$

Example 5: A software company has fixed costs of \$100,000 per month and variable costs of \$10 per user. They charge a subscription fee of \$25 per user per month. How many subscribers does the company need to break even?

Let x be the number of subscribers. ∴ Revenue = $25x$

Total Cost $= 100000 + 10x$

At break-even: $25x = 100000 + 10x \Rightarrow 15x = 100000 \Rightarrow x = 6{,}666.67\ subscribers$

Example 6: Given the marginal revenue function $MR(q) = 100 - 0.4q$ and the marginal cost function $MC(q) = 20 + 0.1q$, find the profit-maximizing quantity and the maximum profit. Fixed costs are \$2000.

At profit-maximizing point, MR = MC:

$$100 - 0.4q = 20 + 0.1q \Rightarrow 80 = 0.5q \Rightarrow q = 160\ units$$

$$TR = \int (100 - 0.4q)dq = 100q - 0.2q^2$$

$$TC = \int (20 + 0.1q)dq + 2000 = 20q + 0.05q^2 + 2000$$

Profit = TR - TC= $(100q - 0.2q^2) - (20q + 0.05q^2 + 2000) = 80q - 0.25q^2 - 2000$

Maximum Profit $= 80(160) - 0.25(160)^2 - 2000 = \$6{,}400$

Example 7: The income distribution is described by the function $f(x) = 2x$, where x is the proportion of the population and f(x) is the proportion of total income earned by the bottom x of the population. Calculate the Gini coefficient.

Here, the Lorenz Curve:$L(x) = \int_0^x 2\,tdt = [t^2]_0^x = x^2$

Gini Coefficient:

$$= 1 - 2\int_0^1 L(x)\, dx$$
$$= 1 - 2\int_0^1 x^2\ dx$$
$$= 1 - 2\left[\frac{x^3}{3}\right]_0^1$$
$$= 1 - 2\ \frac{1}{3}$$
$$= \frac{1}{3} \approx 0.333$$

Example 8: A manufacturer's inventory depletes at a rate of $200 - 5t$ units per week, where t is the number of weeks since production. The holding cost is $1 per unit per week, and the setup cost for each production run is $500. If the manufacturer produces every 8 weeks, what is the total inventory cost per cycle?

Inventory level at time t: $I(t) = \int (200 - 5t)\ dt = 200t - 2.5t^2$
Total holding cost per cycle:

$$HC = 1\int_0^8 (200t - 2.5t^2)\ dt = \left[100t^2 - \frac{5t^3}{6}\right]_0^8 = 6400 - \frac{1280}{3} = 5973.33$$

$$\text{Total cost per cycle} = \text{Holding cost} + \text{Setup cost}$$
$$= 5973.33 + 500 = 6{,}473.33$$

12.9 Exercises and Problems

1. Given the demand function $P = 100 - 2Q$ and the supply function$P = 20 + Q$, where P is price and Q is quantity, calculate the consumer and producer surplus when the market is in equilibrium.

2. If a price ceiling of $50 is imposed in the market described in previous question, calculate the new consumer and producer surplus.

3. Calculate the present value of a continuous cash flow of $5000 per year for 10 years, with a continuous interest rate of 6%.

4. A company expects its revenue to grow continuously according to the function $R(t) = 100000e^{0.05t}$, where t is in years. Calculate the present value of the revenue stream for the next 5 years if the discount rate is 8%.

5. A company has a continuous cost function $C(q) = 1000 + 20q + 0.1q^2$ and a revenue function $R(q) = 50q - 0.05q^2$, where q is the quantity produced and sold.

    a. Find the break-even quantity.

   b. Calculate the total revenue and total cost at the break-even point.
6. Given the marginal cost function $MC(q) = 10 + 0.2q$ and fixed costs of $1000:
   a. Find the total cost function.
   b. If the marginal revenue function is $MR(q) = 50 - 0.4q$, find the profit-maximizing quantity and maximum profit.
7. The Lorenz curve for a certain population is given by $L(x) = x^3$, where x is the cumulative proportion of the population and L(x) is the cumulative proportion of income.
   a. Calculate the Gini coefficient for this population.
   b. Interpret what this Gini coefficient means in terms of income inequality.
8. A company faces a constant demand of 1000 units per year. The holding cost is $2 per unit per year, and the fixed cost per order is $50.
   a. Derive an expression for the total annual cost as a function of the order quantity Q.

   b. Find the optimal order quantity that minimizes the total annual cost.

9. Short Answer Questions

   a. Explain how integration is used in calculating consumer and producer surplus.

   b. Describe how the concept of present value relates to integration.

   c. How can integration be applied to inventory management problems?

10. Multiple Choice Questions

    a. In economics, the area between the demand curve and the price line represents:

       i. Producer surplus

       ii. Consumer surplus

       iii. Total revenue

       iv. Profit

    b. The present value of a continuous cash flow is always:

       i. Greater than the future value

       ii. Less than the future value

iii. Equal to the future value

iv. It depends on the interest rate

c. The Gini coefficient is a measure of:

i. Economic growth

ii. Inflation

iii. Income inequality

iv. Productivity

UNIT-13 MATHEMATICS OF FINANCE

13.1 Introduction:

Mathematics is a very old subject from the ancient times and useful for each and every field. The contribution of mathematics is immense in each and every field because the foundation or basic principles are the part of it. The Development of decimal system, zero, and algebra are the major and significant contributions of ancient India. Some Indian mathematicians like Ramanujan made significant contribution in number theory, elliptic curves and modular forms. Chandrasekhara Venkataraman also made significant contribution on scattering of light. Kannan Soundararajan also made significant contribution in the field of number theory, algebraic geometry and mathematics physics and Vijay Kumar has contributed in algebraic geometrics and commutative algebra and mathematics physics. Finance is a modern branch or subject which deals with the money matter. In simple words, finance is related with the application and uses of money for the benefits of the individuals or company. The mathematics of finance is a mixed branch of mathematics and finance which deals with the application of mathematical techniques into the finance for problems solving or optimizing resources. It includes the use of mathematical models and theory to analyze and understand financial systems, flow of finances, make predictions, and optimize financial decisions. Time Value of Money, Interest Rates, Annuities, Investments, Risk Management, Options and Futures, and Financial Modeling are the most useful techniques used in solving the finance related problems.

13.2 Concepts or models of Mathematics of Finance:

Some important concepts or models of mathematics of finance are as under:

1. Time Value of Money: it is a concept that explains how the value of money changes according to the times changes. In simple words, we can say that money received todays is worth more than received in future time. This is called the present and future value of money which consist discounting and compounding

and amortization. Discounting means present value of a future cashflow whereas compounding means future value of a Present money or cashflow. Amortization means paying off a debt or loan. Time value of money has major four elements. i.e., Present Value (PV), Future Value (FV), Time periods, and interest rates.

2. Annuities: it is a stream of income for a specified periods or long-term periods in exchange of a series of payments. Calculating the present and future value of annuities, including fixed and variable annuity, Indexed annuity, deferred annuity and immediate annuity.
3. Investments: with the use of many mathematical concepts one can Analysis the different- different investments and opportunity including stocks, bonds, and portfolios. mathematical models will useful in cost benefit analysis of the investment portfolio and useful for long term decision making.
4. Risk Management: Risk is defined as an uncertainty of future events and possibility of loss due to these future events. Mathematical models helpful in Understanding and managing various types of risk associate with business entity. it includes risk retention policy. Risk diversification policy and risk controlling strategy.
5. Derivatives contract: Options, forward, swaps and Futures all are called as a type of derivatives contract. Mathematical model will be used for Calculating the value of future, forward, swaps and options contracts. The value of derivatives contract is calculated by spot price and market price.
6. Financial Modeling: Different types of construction contract, transportation, assignment, selling and distribution related problems can be solved by mathematical modeling to analyze and forecast financial data of companies.

13.3 Scope of Mathematics in Finance:

It is enormous and diverse, including various applications and techniques. These are some areas where mathematics plays a vital role in finance:

1. Quantitative Analysis: Mathematical models and techniques are helpful in analyzing and understanding the financial data and figure. It identifies trends and helpful in making predictions related to uncertain future market behavior on the basis of data collected. It also helpful in the determination of future trends of business, market and demands of the product on the basis of past data. There are many areas of finance where quantitative analysis is very useful in decision making process for managers, executives and tope management.
2. Risk Management: It is defined as the process or techniques of minimize or reduce the loss of possible future events of the business. There are various techniques of Mathematical such as probability and probability distribution are useful to assess and manage all types of risk including credit risk, market risk, financial risk and operational risk. Mathematical models are useful in assessing, analyzing and measuring the various types of risk.
3. Options and Derivatives: Mathematical models such as the Black-Scholes model is used to calculate price and value of options, futures, forward and swaps. In

modern days, there are numbers of stock brokers and agents used several types of mathematics model for assessing and measuring the value of all the types of derivatives.

4. Portfolio Optimization: Mathematical models or tools such as mean, standard deviation and linear programming are used to optimize investment portfolios and maximize returns. The main objective of using mathematics models is minimize the risk and get a better return from the portfolio.
5. Actuarial Science: techniques or tools of Mathematics is used to calculate insurance policy premiums and assess risk in insurance companies. it is also helpful in determining the installment of loan.
6. Financial Modeling: Mathematical models are used to forecast financial performance of the entity and estimate future cash inflows and outflows from various projects. It also helpful in evaluating investment opportunities thorough using various models of capital budgeting.
7. Machine Learning and Artificial Intelligence: it is a new techniques and future of using Mathematical technique such as neural networks and regression analysis are used to develop predictive models and automate financial decision-making.
8. Econometrics: Mathematical techniques and econometrics involves regression analysis and time series analysis and cyclical's analysis which are used to understand economic trends and forecast economic outcomes of the business.
9. Financial Engineering: Financial engineering is a branch of Mathematics which is used to design and develop new financial products and functions and includes the structured products and exotic derivatives.

13.4 Definition of interest and some other related terms:

Here we can discuss definition of interest and some other related terms in detail.

Definition and meaning of Interest: it is the amount paid by a borrower for the use of a lender's money. If anyone lends some money called lender to a person who borrow money called borrow for a particular period. When borrower would pay and lender receive a more money than the initial borrowing and this excess money paid by borrower and received by lender is called interest. Assume that someone borrow and someone lend it to borrower ₹ 10,000 for a year and pay 15,000 to lender after one year the difference between the borrowing 10,000 and repayment 15,000 i.e., ₹ 5,000 is the amount of interest paid by borrower o lender.

Principal amount: Principal is initial value of borrowing by a borrower from a lender. If someone lend some amount of money to someone called borrower for a fixed tenure that amount of money is called principal or the value of initial investment in any project is also called principal. Suppose someone borrow ₹ 10,000 from a person for fixed period is the amount 10,000 is called Principal. let's we can understand with other illustrations suppose someone deposit 10,000 in own bank account for one year. Here, 10,000 is the principal amount.

Rate of Interest: The rate at which the interest is charged on a principal amount which is borrowed by borrower for a defined length of time. It is defined with I. it is a reward for lender who lends money and expenses of borrower for using principal amount generally on a yearly basis is known to be the rate of interest. The Rate of interest is expressed as percentages. Let's take an illustration invest or deposits 10,000 in own bank account for one year with the interest rate of 5% per annum. It means that amount of interest would earn 500 as interest on principal amount in a year.

Accumulated amount or Balance: Accumulated amount is the total amount or end amount of an investment. It includes the sum total of the initial investment and interest earned. Let's take an illustration, suppose someone deposit 10,000 in own bank account for one year with an interest rate of 5% per annum then a person would earn interest of 500 after one year. After one year person will get 10,500 inclusion or addition of principal amount and interest amount is also known as the balance.

13.5 Components or Factor of Interest:

There are many factors or components of interest which is the expenses of the use of their money. There is a various component which will discuss in details.

1. Time value of money: Time value of money means the value of a one unit of money is different in different time periods. Due to inflation the value of money is decreased day by day. The value of money received in future is less than what the value holds today. In other words, the present worth of money received after some time will be less than a money received today. Since a money received today has more value in compare to future receipts.
2. Opportunity Cost: it is known as a choice between using money in different proposal or investments If he invest in anyone proposal then forgoes the return from all others proposal. In other words, the lender has an opportunity cost due to the other possible proposal which can uses for lending same money.
3. Inflation: it is known as the economies term which means a fall in the purchasing power of money or decrease in the value of money. Because of inflation a given amount of money buys lesser goods in the future than what it can buy in present. The borrower wants to compensate the lender for inflation due to this, value of money is decrease in same specific percentage.
4. Liquidity Preference: investor or People have their resources available in a form of cash or balances in accounts that can immediately be converted into investment rather than a cash form because it generates returns. And investment takes some time to realize in cash or money.
5. Risk Factor. There is always a risk involves with the investment that if the borrower will go bankrupt or default on the loan. Risk is a determinable on the basis of many factors such as principal amount and time. The rate of interest is the elements that attract the investor more. A lender generally charges more interest rate for taking more risk.

13.6 Simple interest and Compound interest:

Simple interest and Compound interest is the method of computing interest. Interest accrues as either simple interest or compound interest. Let's discuss in details the simple interest:

13.6.1 Simple Interest: what is simple interest?

It is the methodology of computing interest and accumulated amount for an investment or principal amount with a simple rate over a period of time. As we already know the money or amount that borrower borrowed from the lender is known as principal and the additional amount of money that borrower have to pay for using lenders money is known as interest. The specific rate at which interest was paid by borrower to lender for keeping 10,000 for one year is known as the percent rate of interest per annum. Let's say that if the money is borrowed at the rate of 10% per annum, then the interest paid for keeping 10,000 for one year is 1000. The sum of principal and interest is known as the total amount.

The amount of interest pay is comparable to the money that borrowed by borrower and related to the period of time. When the tenure of money for which money kept is more than the amount of interest is also increased. The money, the time, and the interest are also proportionate to the rate of interest agreed by the lenders and the borrower. Thus, we can say that interest varies directly with principal, time and rate

Simple interest is the amount of interest or money computed on the principal for the entire period of borrowing. It is calculated on the outstanding principal balance left with borrower after paying some installment and not on interest previously earned. It means no interest is paid on interest earned during the term of loan. In simple words there is not reinvestment of interest earned by lenders and simple interest is calculated on outstanding amount of principal.

Simple interest can be computed through the following formulas:

$S.\,I = (P \times R \times T) / 100$

Here, I = Denotes amount of interest

P = Principal (value of an investment)

T = Time in years

R = Annual interest rate in decimal

$A = P + I$

Here, A = Accumulated amount (final value of an investment) = principal plus interest

We can make more formulas through above two formulas

$= P + PRT$

= P (1 + R/100 x T)

I = P - A

Let us consider use the following formulas in some illustrations:

Illustration 1: How much interest will be earned on 20,000 at 10% simple interest for 1 year?

Answers: in this illustration, principal amount or initial investment is 20,000, rate of interest = 10% or 10/100 = 0.10 and time is 1 year and amount of interest = ?

I = (P x R x T) / 100

Here, I = Denotes amount of interest

P = Principal (value of an investment)

T = Time in years

R = Annual interest rate in decimal

= 20,000 x 0.10 x 1 = 2,000

Illustration 2: Ankit deposited ₹500,000 in a bank for three years with the interest rate of 7.5% p.a. How much interest would earn?

Answers: in this illustration, principal amount or initial investment is 500,000, rate of interest = 7.5% or 7.5/100 = .075 and time is 3 year and amount of interest = ?

I = P x R x T

Here, I = Denotes amount of interest

P = Principal (value of an investment)

T = Time in years

R = Annual interest rate in decimal

= 5,00,000 x 0.075 x 3 = 1,12,500

Illustration 3: Ankit deposited ₹500,000 in a bank for three years with the interest rate of 7.5% p.a. Find out the How much interest would earn and what will be the final value of investment?

Answers: in this illustration, principal amount or initial investment is 500,000, rate of interest = 7.5% or 7.5/100 = .075 and time is 3 year and amount of interest = ? Final value of investment = ?

I = P x R x T

Here, I = Denotes amount of interest

P = Principal (value of an investment)

T = Time in years

R = Annual interest rate in decimal

= 5,00,000 x 0.075 x 3 = 1,12,500

Formula of final value of investment A = P (1 + R/100 x T)

= 5,00,000 x (1 + 7.5/100 x 3)

= 5,00,000 x (1 + 0.225)

= 5,00,000 x (1.225)

= 6,12,500

We can find out the value of investment through other formula also

A = P + I

= 5,00,000 + 1,12,500 = 6,12,500

Illustration 4: Kalu deposited ₹ 3,00,000 in bank for 2 years at simple interest rate of 8%. Find out How much interest would he earn? And How much would be the final value of deposit?

Answers: in this illustration, principal amount or initial investment is 300,000, rate of interest = 8% or = 0.08 and time is 2 year and amount of interest = ? Final value of investment = ?

Amount of interest (I) = P x R/100 x T

I = 3,00,000 x 8 / 100 x 2 = 48,000

Final value of deposit (A) = P + I

= 3,00,000 + 48,000 = 3,48,000

Illustration 5: Find the rate of interest if the amount owed after 6 months is ₹ 1050, borrowed amount being 1000.

Answers: in this illustration, principal amount or initial investment is 1,000, rate of interest = ? and time is 0.5 year and Final value of investment = 1050

Formula of Final value of investment **(**A) = P + P x R/100 x T

1050 = 1000 + 1000 x R/100 x 0.5

50 = 500 x R/100

R = 50 / 500 = 10%

Illustration 6: Ram invested ₹ 70,000 in a bank at the rate of 6.5% p.a. with simple interest rate and received 85,925 after the end of term. Find out the period for which sum was invested by Ram.

Answers: in this illustration, principal amount or initial investment is 70,000, rate of interest = 6.5 % p.a. and time = ? and Final value of investment = 85,925

Final value of investment **(**A) = P x (1 + R/100 x T)

85,925 = 70,000 x (1 + 6.5 / 100 x T)

85,925 / 70,000 = 100 + 6.5T /100

85,925 x 100 / 70,000 – 100 = 6.5T

22.75 = 6.5T

T = 3.5 years

Illustration 7: Kalpesh deposited some amount of money in a bank for 7.5 years at the rate of 6% p.a. with simple interest rate option and received 1,01,500 at the end of the term. Compute initial deposit or principal amount.

Answers: in this illustration, principal amount or initial investment = ?, rate of interest = 6 % p.a. and time is 7.5 years and Final value of investment = 1,01,500

Final value of investment (A) = P x (1 + R/100 x T)

1,01,500 = P x (1 + 6 / 100 x 7.5)

1,01,500 = P x (1 + 45 / 100)

1,01,500 = P x (145 / 100)

P = (1,01,500 x 100) /145 =70,000

Initial deposit of Kapil = 70,000

Illustration 8: A sum of 46,875 was lent out at simple interest and at the end of 1 year 8 months the total amount was 50,000. Find the rate of interest.

Answers: in this illustration, principal amount or initial investment = 46,875, rate of interest = ? and time is 1 year and 8 months and Final value of investment = 50,000

The formula of Final value of investment (A) = P x (1 + R/100 x T)

= 50,000 = 46,875 x (1 + R/100 x 1.67)

= 50,000 / 46,875 = 1 + 1.67 x T

= (1.067 - 1) / 1.67 = R/100

R = 0.04, rate of interest = 4%

Illustration 9: What sum of money will produce 28,600 as an interest rate of 2.5% in 3 years and 3 months at p.a. with simple interest?

Answers: in this illustration, principal amount or initial investment = ?, rate of interest = 2.5% p.a. and time is 3 year and 3 months and Final value of investment = 28,600

The formula of amount of interest (I) = P x R/100 x T

= 28,600 = P x 2.5/100 x 3.25

= 28,600 = 8.125P /100

= 28,600 x 100 = 8.125P

P = (28,60,000) / 8.125 = 3,52,000

3,52,000 will produce 28,600 interests in 3 years and 3 months at 2.5% p.a. simple interest

Illustration 10: In what time will ₹ 85,000 amount to ₹ 1,57,675 at 4.5% p.a.?

Answers: in this illustration, principal amount or initial investment = 85,000, rate of interest = 4.5% p.a. and time =? and Final value of investment = 1,57,675.

The formula of A = P x (1 + R/100 x T)

1,57,675 = 85,000 x (1 + 4.5 / 100 x T)

1,57,675 / 85000 = (100 + 4.5t) /100

1.855 x 100 = 100 + 4.5T

185.5 -100 = 4.5T

T = 85.5 / 4.5 = 19

In 19 years 85,000 will be ₹1,57,675 at 4.5% p.a.

13.6.2 Compound Interest:

Simple interest is known when the principal remains the same for the entire period or time. Simple interest is the amount of interest which is constant or same. However, in the practical method which applied by the banks, insurance corporations and other money lending and deposit companies' calculation of interest is different. To understand this method, we consider an illustration- Suppose someone deposits 50,000 in AXIS bank for 2 years at 7% p.a. at compounded interest annually. Interest will be calculated in the following way:

I = P x R/100 x T

Interest for first year

I = 50,000 x 7 / 100 x 1 = 3,500

For calculating interest for second year principal would not be the initial deposit or same which is considering in first year. Principal for calculating interest for second year will be the initial deposit plus interest for the first year. Therefore, principal for calculating interest for second year would be

= 50,000 +3,500 (interest of first year) = 53,500

I = P x R/100 x T = 53,500 x 7 / 100 x 1= 3,745

For calculating interest for second year principal would not be the initial deposit. Principal for calculating interest for second year will be the initial deposit plus interest for the first year. Therefore, principal for calculating interest for second year would be

Total interest = Interest for first year and Interest for second year = (3,500 + 3,745) = 7,245

This interest as per compound interest 245 more than the simple interest. We noticed that the excess in interest is because of the principal for the second year which is more than the principal of first year. These types of interest calculation are known as compound interest.

The compound interest as the interest that accrues when the earnings /interest of every year is added to the principal thus increasing the principal base on which subsequent interest is computed.

Example 11: Sakina deposited 3,00,000 in a nationalized bank for three years. If the rate of interest is 10% p.a., calculate the interest that bank has to pay to Sakina after three years if interest is compounded annually.

Answers: Principal for first year 3,00,000

Interest for first year = P x R/100 x T

= 3,00,000 x 10 / 100 x 1 = 30,000

Principal for the second year = Principal for first year + Interest for first year

= 3,00,000 + 30,000 = 3,30,000

Interest for second year = 3,30,000 x 10 / 100 *1 = 33,000

= Principal for the third year = Principal for second year + Interest for second year

=3,30,000 + 33,000 = 3,63,000

Interest for the third year = 3,63,000 x 10 / 100 x 1 = 36,300

Compound interest at the end of third year = (30,000 + 33,000 + 36,300) = 99,300

Amount at the end of third year = Principal (initial deposit) + compound interest = 3,00,000 + 99,300 = 3,99,300

Formula for compound interest

Taking the principal as P, the rate of interest as I, the time period as n, the accrued amount after n payment periods as A_n and accrued amount at the end of first payment period

A1 = P + P x R/100 = P(1 + R/100)

at the end of second payment period (A2) = A1 + A1 x R/100 = A1 (1 + R/100)

$= P (1 + R/100) \times (1 + R/100)$

$= P \times (1 + R/100)^{2}$

at the end of third payment period (A3) = A2 + A2 x R/100 = A2 (1 + R/100)

$= P (1 + R/100)^{2} \times (1 + R/100)$

$= P (1 + R/100)^{3}$

A n = A n - 1 + A n-1 x R/100

=A n - 1 (1+R/100)

$=P (1 + R/100)^{n-1}(1+R/100)$

= P (1 + R/100) n

Thus, the accrued amount An

$An = P (1 + R/100)^{n}$

$Interest = An - P = P (1 + R/100)^{n} - P$

$= P [(1 + R/100)^{n} - 1]$

Illustration 12: 2,000 is invested at annual rate of interest of 10%. What is the amount after two years if compounding is (a) Annually (b) half yearly (c) Quarterly.

Answers: (a) when Compounding is annually

principal P = 2,000, interest is compounded yearly the number of conversion periods n in 2 years and the rate of interest per conversion period i = 0.10

The formula of accrued amount $An = P (1 + R/100)^{n}$

$A2 = 2000 \times (1 + 0.1)^{2}$

$=2000 \times (1.1)^{2}$

= 2000 x 1.21 = 2,420

(b) when compounding is half yearly n = 2 x 2 = 4 and R = 0.1/2 = 0.05

$A4 = 2000 \times (1 + 0.05)^{4}$

=2000 x (1.215)

= 2000 x 1.21 = 2,431

(c) when compounding is quarterly n = 4 x 2 = 8 and R = 0.1/4 = 0.025

$A8 = 2000 \times (1 + 0.025)^{8}$

= 2000 x 1.218 = 2,436.80

Illustration 13: Determine the compound amount and compound interest on 1000 at 6% compounded half yearly for 6 years.

Answers: R = 0.06 / 2 = 0.03

P = 1000

n = 6 x 2 = 12

Compound Amount A12 = P x $(1 + R/100)^{n}$

= 1000 x $(1 + 0.03)^{12}$

= 1000 x 1.42 = 1.426

Compound Interest = (1,425.76 – 1,000) = 426

Illustrations 14: On what sum will the compound interest at 5% per annum for two years compounded annually be ₹ 1,640?

Answers: the interest is compounded yearly the number of conversion periods = 2 and the rate of interest per conversion period = 5%. So, n = 2 and R = 0.05

Formula of Compound Interest = P $[(1 + R/100)^{n} - 1]$

= 1640 = P $[(1 + 0.05)^{2} - 1]$

= 1640 = P [1.1025 - 1]

P = 1640 / 0.1025 = 16,000

Required sum P = 16,000

Illustration 15: What annual rate of interest compounded annually doubles an investment in 7 years?

Answers: If the principal P then An = 2P

An = P $(1 + R/100)^{n}$

2P = P $(1 + R/100)^{7}$

$2^{1/7}$ = (1 + R/100)

1.104 = 1 + R/100

R = 0.104

Required rate of interest = 10.41% p.a.,

13.7 Difference between simple interest and compound interest:

The difference between simple interest and compound interest is that in simple interest the principal remains constant throughout on the other side of compound interest principal goes on changing at the end of every year. For a given principal, rate and time the compound interest is generally more than the simple interest

13.8 Effective rate of interest:

If interest is compounded for more than one in a year then the effective interest rate for a next exceeds the per annum interest rate. Suppose anyone invest ₹ 1,000 for a first year at the rate of 6% per annum and interest is compounded as half yearly and quarterly then the Effective interest rate for a year will be more than 6% per annum. Let us understand the interest.

Interest for first 6 months = 10,000 × 6/100 x 0.5 = 300

Principal for calculation of interest for next 6 months = Principal for first period + Interest for first six months

= (10,000 + 300) = 10,300

Interest for next six months = 10,300 × 6/100 × 0.5 = 309

Total interest earned during the whole year = Interest for first 6 months + Interest for next 6 months

= (300 + 309) = ₹609

Now, verify the rate of Interest with amount of interest

I = P x E x T

Where, I = Amount of interest

E=Effective rate of interest in decimal

T = Time period

E = 609 / 10,000 = 0.609 = 6.09 %

13.9 Present value and Future value:

Future value: Future value is the value of cash of an investment at future period. In simple words, it is tomorrow's value of today's money. We already know the concept of simple and compound interest and various examples of it.

The formula of future value of any single rupees $An = P \times (1 + r)^n$

Where A = Accumulated amount

n = number of conversion period

r = rate of interest per conversion period in decimal

P = principal

We can replace the value of A by future value (F) and P by single cash flow (CF)

$F = CF \times (1 + r)^n$

Illustration 17: You invest 3000 in a two year investment that pays you 12% per annum. Calculate the future value of the investment

Answers: $F = CF \times (1 + r)^n$

F = Future Cash flows

C.F =Cash flow = 3,000

n = time period = 2

r = rate of interest = 0.12

$F = 3{,}000 \times (1 + 0.12)^2$

= 3,000 x 1.254 = 3,763

Present value: The future value is tomorrow's value of today's money compounded at some interest rate and present value is today's value of tomorrow's money discounted at the interest rate. Future value and present value are the reciprocal of each other. the present value of a cash flow can get by applying compound interest formula.

The present value P of the amount A, due at the end of n period at the rate of i per interest period can get through the following formula $A = P \times (1 + r)^n$

The formula of present value $P = A^n / (1 + r)^n$

Illustration 18: What is the present value of 1 to be received after two years compounded annually at 10% interest rate?

Answers: A = 1, i = 10% = 0.10

Present value $P = A^n / (1 + r)^n$

$= 1 / (1 + 0.1)^2$

= 1 / 1.21

= 0.826 = 0.83

Illustration 19: Find the present value of 10,000 to be required after 5 years if the interest rate be 9%

Answers: A = 10,000, r = 9% = 0.09

Present value $P = A^n / (1 + r)^n$

$= 10{,}000 / (1 + 0.09)^5$

= 10,000 / 1.538

= 6,499

❖ **Exercises**

1. MCQs

1. What is the formula of Simple Interest?

   A) (P + R + T) / 100

   B) (P x R x T) / 100

   C) (P - R - T) / 100

   D) None of these

   Answer: B) (P x R x T) / 100

2. What is the formula of Compound Interest?

   A) P x (1 - R/ 100) - P

   B) P x $(1 - R/ 100)^T$ - P

   C) P x $(1 + R/ 100)^T$ - P

   D) None of these

   Answer: C) P x $(1 + R/ 100)^T$ – P

3. Which type of interest is calculated on the initial principal amount?

   A) Compound Interest

   B) Simple Interest

   C) Both A and B

   D) None of the above

   Answer: B) Simple Interest

4. Which type of interest is calculated on the initial principal amount and accrued interest?

   A) Compound Interest

   B) Simple Interest

   C) Both A and B

   D) None of the above

   Answer: A) Compound Interest

2. Answer the following shorts and long questions:

1. Explain the Concepts or Models of Mathematics of Finance.

2. Explain the Scope of Mathematics in Finance.
3. Explain the Definition of interest.
4. Explain the Components or Factors of Interest.
5. Explain the Simple interest and Compound interest.
6. Explain the Difference between simple and compound interest.
7. Explain the Effective rate of interest.
8. Explain the Present value and Future value.

UNIT-14 ANNUITIES

14.1 Introduction

14.2 Future Value Of Immediate Annuity

14.3 Future Value Of Annuity Due

14.4 Present Value Of Immediate Annuity

14.5 Present Value Of Annuity Due

14.6 Applications

14.1 Introduction

In the chapter, we studied how to find the present value of a payment due after a given interval or period at a given rate of interest and also how to find the accumulated value of a sum now invested at the end of given time.

In the present chapter, we shall consider how to find the present value of a series of payments at given rate of interest payable during a given period as at the beginning of the period as also the accumulated value of a series of payments during the given period as at the end of period.

A series of payments made at successive periods (intervals) of time is called an **annuity**. If all the payments are equal, then the annuity may be called a level or **uniform annuity**, otherwise it may be called a **variable annuity**. If the total interval for which the payments of the annuity are to be made is a definite number of years, not depending upon any contingency, then the annuity is called **annuity certain**. If the payments are to be made during the life of a person it is called a **life annuity**.

An annuity is usually referred to as so much per annum even when the frequency of payment is different from yearly. Thus if a payment of Rs. 100 falls due every year the annuity is of Rs.100 p.a. (**per annum/per year**), while if a payment of Rs. 25 falls due every quarter, we call it an annuity of Rs.100 p.a. payable quarterly or an annuity with annual rate of Rs.100 payable quarterly. Where the frequency is not specified it is understood to be yearly.

If the successive payments of the annuity are made at the **end of** the successive periods, then the annuity is called an **immediate annuity**. If these successive payments are made at the **beginning of** each successive period, then the annuity is called an **annuity-due**. In the case of an immediate annuity certain of Re.1 p.a. for 10 years there are ten yearly payments in all, the first payment of Re.1 being made at the end of the first year and the last at the end of the 10^{th} year.
In the case of an annuity certain due of Re.1 p.a. for 10 years, there are again 10 payments, but the first payment is made at the beginning of the first year and the 10th payment is made at the beginning of the 10th year i.e. at the end of 9 years.

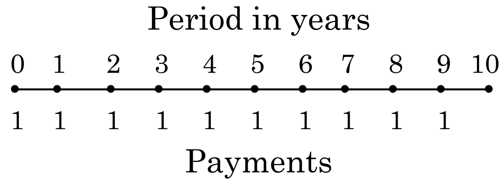
Immediate Annuity of 1 p.a. for 10 years

Period in years

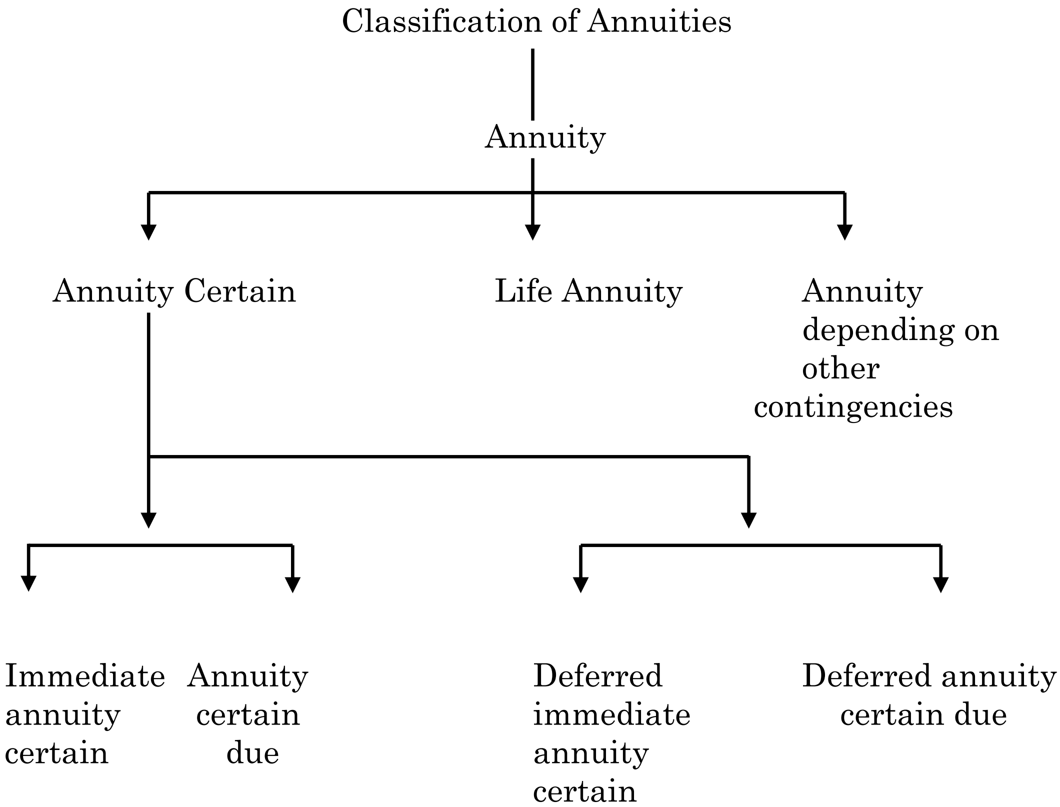
0	1	2	3	4	5	6	7	8	9	10
	1	1	1	1	1	1	1	1	1	1

Payments

Annuity Due of 1 p.a. for 10 years



The following chart shows various types of annuities :



It may be noticed that in the case of an immediate annuity of Re.1 p.a. payable more frequently in a year, say quarterly, the periodical payments are 1/4every quarter and the first payment is at the end of the first quarter and the last quarterly payment at the end of the *last* quarter i.e. at the end of the period of the annuity. For a similar annuity due, the first quarterly payment of 1/4is made at the beginning of the first quarter and the last quarterly payment at the beginning of the last quarter.

14.2 Future Value Of Immediate Annuity

We shall now find an equation for the accumulated value or amount of an immediate annuity of Re.1 p.a. for n years. The accumulated value of an immediate annuity of Re.1 p.a. for n years at rate of interest of i per unit per annum is denoted by the symbol A. The procedure is similar to that for finding the present value of an annuity i.e., we accumulate each individual payment at the end of n years and add up the accumulated values.

Thus at the end of the period of n years the accumulated value of the last payment due = Re.1.

The accumulated value of the last payment but one = $(1 + i)$

The accumulated value of the last payment but two = $(1 + i)^2$
Lastly, the accumulated value of the first payment = $(1 + i)^{n-1}$

Adding we get,

Accumulated Value = A = 1 + (1 + i) + (1 + i)^2 + ... + (1 + i)$^{n-1}$

The above expression is a geometric progression whose 1st term is 1 with common ratio (1 + i) and number of terms n.

$$A = \frac{(1+i)^n - 1}{i}$$

If the annuity is not Re.1 p.a. but a p.a., the accumulated value of the annuity is

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

Suppose a constant sum of Re. 1 is deposited in a savings account at the end of each year for four years at 5% interest. This implies that Re. 1 deposited at the end of the first year will grow for three years, Re. 1 at the end of second year for 2 years, Re.1 at the end of the third year for one year and Re.1 at the end of the fourth year will not yield any interest. Using the concept of compound interest we can compute the future value of annuity. The compound value (compound amount) of Re.1 deposited in the first year will be

A_3 = Rs. 1 (1 + 0.05)^3
= Rs. 1.1576

The compound value of Re.1 deposited in the second year will be

A_2 = Rs. 1 (1 + 0.05)^2= Rs. 1.1025

The compound value of Re.1 deposited in the third year will be

A_1 = Rs. 1 (1 + 0.05)^1
= Rs. 1.05

The compound value of Re.1 deposited at the end of fourth year will remain Re. 1.

The aggregate compound value of Re. 1 deposited at the end of each year for four years would be:

Rs. (1. 1576 + 1.1025 + 1.050 + 1.00) = Rs. 4.3101

This is the compound value of an annuity of Re.1 for four years at 5% rate of interest. The above computation is summarized in the following table:

End of year	**Amount Deposit (Re.)**	**Future value at the end of fourth year (Rs.)**
0	-	-
1	1	1(1 + 0.05)^3 = 1.1576
2	1	1(1 + 0.05^2 = 1.1025
3	1	1(1 + 0.05)^1 = 1.050
4	1	1(1 + 0.05)^0 = 1
	Future Value	4.3101

- **The future of annuity when annuity is paid once in a year at the end of each period.**

Ex.1 Mr. Vicky deposits Rs. 10,000 at the end of every year with Edelweiss Finance at 14% rate of interest. Find the sum that he will receive from Edelweiss Finance at the end of 10 years.

Solution:

Here, a = 10,000 ; $i = \frac{R}{100} = \frac{14}{100}$ =0.14; n = 10; Future value A = ?

$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$

$A = \frac{10,000}{0.14}\left\{(1+0.14)^{10} - 1\right\}$

$A = \frac{10,000}{0.14}\left\{(1.14)^{10} - 1\right\}$

$A = 71428.57 \times (3.7072 - 1)$

$\therefore A$ = 1,93,371.42

Thus, Mr. Vicky will receive Rs.1,93,371.42.

Ex.2 Vishal deposits Rs.2,500 at the end of every year at 15% rate of compound interest with a financer (NBFC). Find out what amount he would receive at the end of 20 years from that financer.

Solution :

Here, a = 2,500; $i = \frac{R}{100} = \frac{15}{100}$ =0.15; n = 20; Future value A = ?

$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$

$A = \frac{2,500}{0.15}\left\{(1+0.15)^{20} - 1\right\}$

$A = \frac{2,500}{0.15}\left\{(1.15)^{20} - 1\right\}$

A = 16,666.67 $\times$ (16.3665 – 1)

A = 2,56,108.38

Evaluation of $(1.15)^{20}$:

$x = (1.15)^{20}$

$\log x = 20 \log 1.15$

= 20 (0.0607) = 1.214

$\therefore x$ = antilog 1.214 = 16.3665

Thus, Vishal will receive Rs.2,56,108.38.

Ex.3 Mr. Suresh purchased a mobile costing Rs. 60,000 on 1-1-2024. Its expected life is 8 years. It is assumed that after 8 years, the cost of the same mobile will be increased by 25%. What sum should he deposit on 31st December of every year at 15% rate of interest so that enough sum is accumulated to meet the cost of new mobile?

Solution: Cost of computer after 8 years = Rs. 60,000 + 25 % of Rs. 60,000

= Rs. 60,000 +Rs.15,000

= Rs. 75,000

Here, A = 75,000; n = 8; $\frac{R}{100} = \frac{15}{100}$ = 0.15; Annuity a = ?.

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

$$75{,}000 = \frac{a}{0.15}\left\{(1+0.15)^8 - 1\right\}$$

$$75{,}000 = \frac{a}{0.15}\left\{(1.15)^8 - 1\right\}$$

$75{,}000 \times 0.15 = a\,(3.059 - 1)$

$11{,}250 = a\,(2{,}059)$

$$a = \frac{11{,}250}{2.059}$$

a = 5463.81

Thus, the sum to be deposited every year on 31st December is Rs. 5463.81.

- **The future of annuity when annuity is paid more than once in a year at the end of each period.**

If annuity is paid or received more than once in a year at the end of each period:

$$A = \frac{a}{\frac{i}{k}}\left\{\left(1+\frac{i}{k}\right)^{nk} - 1\right\}$$

Where k = number of times annuity is paid or received in a year.

Ex.4 Mr. Mukesh deposits a sum of Rs. 3000 at the end of every month from his salary in his provident fund account. If the rate of interest is 12%, then find the sum accumulated in his provident fund account at the end of his services of 35 years.

Solution :

Here, a = 3,000; $i = \frac{R}{100} = \frac{12}{100}$ =0.12; k = 12 (monthly deposit);

n = 35 Future value of A = ?

$$A = \frac{\mathrm{a}}{\frac{\mathrm{i}}{\mathrm{k}}}\left\{\left(1+\frac{i}{k}\right)^{nk} - 1\right\}$$

$$A = \frac{3{,}000}{\frac{0.12}{12}}\left\{\left(1+\frac{0.12}{12}\right)^{35\times 12} - 1\right\}$$

$A = 3{,}00{,}000\ \{(1+0.01)^{420} - 1\}$ Evaluation of $(1.01)^{420}$

$A = 3{,}00{,}000\ \{(1.01)^{420} - 1\}$ $x = (1.01)^{420}$

A = 3,00,000 (63.97 – 1)   log x = 420 log 1.01

A = 3,00,000 (62.97)   = 420 (0.0043) = 1.806

A = 1,88,91,000 $\therefore$ x = antilog (1.806) = 63.97

Rs. 1,88,91,000 will be accumulated in his provident fund account.

[Note: Using calculator value of $(1.01)^{420}$ = 65.3096 and A = 1,92,92,880]

Ex.5 Biplav pays Rs. 200 at the end of every month toward his provident fund account from his salary. If the rate of compound interest is 12%, then find out the total amount credited in his provident fund account at the end of 30 years.

Solution : Here, a = 200; $i = \frac{R}{100} = \frac{12}{100} = 0.12$; k = 12 (monthly deposit);

n = 30 Future value of A = ?

$$A = \frac{a}{\frac{i}{k}}\left\{\left(1+\frac{i}{k}\right)^{nk} - 1\right\}$$

$$A = \frac{200}{\frac{0.12}{12}}\left\{\left(1+\frac{0.12}{12}\right)^{30\times 12} - 1\right\}$$

A = 20,000 $\{(1 + 0.01)^{360} – 1\}$

A = 20,000 $\{(1.01)^{360} – 1\}$ Evaluation of $(1.01)^{360}$

A = 20,000 (35.32 – 1) $x = (1.01)^{360}$

A = 20,000 (34.32)   log x = 360 log (1.01)

A = 6,86,400   = 360 (0.0043) = 1.548

$\therefore x$ = antilog (1.548) = 35.32

Rs. 6,86,400 will be accumulated in his provident fund account.

[Note: Using calculator value of $(1.01)^{360}$ = 35.9496 and A = 6,98,992]

Ex.6 Mr. Anmol's daughter Tiny is 10 years old. Mr. Anmol deposits certain fixed sum at the end of every month with LIC of India at 18% rate of interest. Tiny will receive Rs. 2,60,000 when her age will be 25 years. Find the sum deposited every month by Mr. Anmol.

Solution:

Here, A = 2,60,000 ; $i = \frac{R}{100} = \frac{18}{100} = 0.18$; n = 15 (25 – 10 years),

k =12 (monthly deposit); Annuity a = ?

$$A = \frac{a}{\frac{i}{k}}\left\{\left(1+\frac{i}{k}\right)^{nk} - 1\right\}$$

$$2,60,000 = \frac{a}{\frac{0.18}{12}} \left\{\left(1+\frac{0.18}{12}\right)^{15\times 12} - 1\right\}$$

$$2,60,000 = \frac{a}{0.015}\{(1 + 0.015)^{180} - 1)\}$$

$$2,60,000 \times 0.015 = a\{(1.015)^{180} - 1\}$$

$$3,900 = a\,(14.19 - 1)$$

$$3,900 = a\,(13.19)$$

$$a = \frac{3,900}{13.19}$$

$$a = 295.67$$

Evaluation of $(1.015)^{180}$:

$$x = (1.015)^{180}$$

$$\log x = 180 \log (1.015)$$

$$= 180(0.0064) = 1.152$$

$$\therefore x = \text{antilog}\,(1.152) = 14.19$$

The sum to be deposited every month is Rs. 295.67.

[Note: Using calculator value of $(1.015)^{180}$= 14.5894 and A = 287.09]

14.3 Future Value Of Annuity Due

As we know that in Annuity due, first receipt or payment is made at the beginning of the period. Annuity regular assumes that the first receipt or the first payment is made at the end of first period. The relationship between the value of an annuity due and an immediate annuity in case of future value is:

Future value of an Annuity due = $(1 + i)$ (Future value of immediate annuity)

Calculating the future value of the annuity due we use the following equation.

$$A = (1 + i)\left(\frac{a}{i}\right)\{(1 + i)^{n} - 1\}$$

➢ **The future value of annuity when the annuity paid or receives once in a year at the beginning of each period.**

Ex.7 Mr. Viraj deposits Rs. 5,000 on 1st of January in the beginning of each year in recurring account at post office; the rate of interest is 9%. What sum will be accumulated in his account at the end 15 years?

Solution : Here, $a = 5,000$; $i = \frac{R}{100} = \frac{9}{100} = 0.09$; $n = 15$; Future value of A = ?

$$A = (1 + i)\left(\frac{a}{i}\right)\{(1 + i)^{n} - 1\}$$

$$A = (1 + 0.09)\left(\frac{5,000}{0.09}\right)\{(1+ 0.09)^{15} - 1\}$$

$$A = (1.09)\left(\frac{5,000}{0.09}\right)\{(1.09)^{15} - 1\}$$

$$A = 1.09 \times \frac{5,000}{0.09}\,(3.6425 - 1)$$

$$A = 1.09 \times \frac{5,000}{0.09} \times 2.6425$$

A = 1,60,018.06

Rs. 1,60,018.06 will be accumulated in Mr. Viraj account.

Ex.8 Mr. Ketan deposits Rs. 9,000 on 1st of January in the beginning of each year in the recurring account at SBI. The rate of interest is 9%. What sum will be accumulated in his account at the each of 15 years?

Solution : Here, a = 9,000; $i = \frac{R}{100} = \frac{9}{100}$ = 0.09; n = 15; Future value of A = ?

$$A = (1 + i)\left(\frac{a}{i}\right)\{(1 + i)^n - 1\}$$

$$A = (1 + 0.09)\left(\frac{9000}{0.09}\right)\{(1 + 0.09)^{15} - 1\}$$

$$A = (1.09)\left(\frac{9,000}{0.09}\right)\{(1.09)^{15} - 1\}$$

$$A = 1.09 \times \left(\frac{9,000}{0.09}\right)(3.6425 - 1)$$

$$A = 1.09 \times \left(\frac{9,000}{0.09}\right) \times 2.6425$$

A = 2,88,032.5

Rs. 2,88,032.5 will be accumulated in Mr. Ketan account.

Ex.9 Mr. Harish's son Rajesh is 10 years old. Mr. Harish deposits certain sum with Axis Bank in the beginning of every year at 14% rate of interest such that his son Rajesh will receive Rs.5,00,000 when he would be 25 years old. Find the sum that Mr. Harish deposits in the beginning of every year.

Solution : Here, A = 5,0,000; $i = \frac{R}{100} = \frac{14}{100}$ = 0.14;

n = 15(25 years – 10 years); Annuity a = ?

$$A = (1 + i)\left(\frac{a}{i}\right)\left\{(1 + i)^n - 1\right\}$$

$$5,00,000 = (1 + 0.14)\left(\frac{a}{0.14}\right)\{(1 + 0.14)^{15} - 1\}$$

$$5,00,000 = (1.14)\left(\frac{a}{0.14}\right)\{(1.14)^{15} - 1\}$$

$$5,00,000 = (1.14)\left(\frac{a}{0.14}\right)(7.1379 - 1)$$

$$5,00,000 = 1.14 \times \frac{a}{0.14} \times 6.1379$$

$$a = \frac{(5,00,000)(0.14)}{(1.14)(6.1379)}$$

a = 10,004

Mr. Harish deposits Rs.10,004 in the beginning of every year.

Ex.10 Shah's son is five-year-old. Shah deposits certain amount, at the beginning of every year at 14% rate of interest such that Shah will receive Rs.3,00,000 when his son would be 20 years old. Find the sum that Shah deposit in the beginning of every year.

Solution : Here, A = 3,0,000; $i = \frac{R}{100} = \frac{14}{100} = 0.14$;

n = 15 (20 years – 5 years); Annuity a = ?

$$A = (1+i)\left(\frac{a}{i}\right)\left\{(1+i)^n - 1\right\}$$

$$3,00,000 = (1 + 0.14)\ \{(1 + 0.14)^{15} - 1\}$$

$$3,00,000 = (1.14)\left(\frac{a}{0.14}\right)\{(1.14)^{15} - 1\}$$

$$3,00,000 = (1.14)\left(\frac{a}{0.14}\right)(7.1379 - 1)$$

$$3,00,000 = 1.14 \times \frac{a}{0.14} \times 6.1379$$

$$a = \frac{(3,00,000)(0.14)}{(1.14)(6.1379)}$$

a = 6,002.39

Shah deposits Rs.6,002.39 in the beginning of every year.

➢ **The future value of annuity when the annuity paid or received more than once in a year at the beginning of each period.**

If annuity is paid or received more than once in a year in the beginning of each period:

$$A = \left(1+\frac{i}{k}\right)\frac{a}{\frac{i}{k}}\left\{\left(1+\frac{i}{k}\right)^{nk} - 1\right\}$$

Ex.11 Mr. Soham opens a Cumulative Time Deposit account in a post office for 60 months. He deposits Rs. 500 at the beginning of every month. If the rate interest is 10 %, what sum would he receive on maturity?

Solution : Here, $a = 500$; $i = \frac{R}{100} = \frac{10}{100} = 0.1$; $n = 60$ month = 5 years;

$k = 12$ (monthly) Future value of A = ?

$$A = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{\left(1+\frac{i}{k}\right)^{nk}-1\right\}$$

$$A = \left(1+\frac{0.1}{12}\right)\left(\frac{500}{\frac{0.1}{12}}\right)\left\{\left(1+\frac{0.1}{12}\right)^{5\times 12}-1\right\}$$

$$A = \left(\frac{12.1}{12}\right)\times\left(\frac{500\times 12}{0.1}\right)\left\{\left(\frac{12.1}{12}\right)^{60}-1\right\}$$

$A = \frac{12.1\times 500}{0.1}\{(1.0083)^{60} - 1\}$

$A = \frac{12.1\times 500}{0.1}\{1.622 - 1\}$

$A = 121\times 500 \times 0.622$

$A = 37{,}631$

Evaluation of $(1.0083)^{60}$:

$x = (1.0083)^{60}$

$\log x = 60 \log (1.0083)$

$= 60(0.0035) = 0.21$

$\therefore x = \text{antilog}(0.21) = 1.622$

Mr. Soham will receive Rs. 37,631 on maturity.

[Note: Using calculator value of $(1.0083)^{60}$= 1.642 and A = 38,841]

Ex.12 Mr. Rachit opens a Cumulative Time Deposit account in a post office for 60 months. He deposits Rs.1,600 at the beginning of every month. If the rate of interest is 10%, what sum would he receive on maturity?

Solution : Here, $a = 1{,}600$; $i = \frac{R}{100} = \frac{10}{100} = 0.1$; $n = 60$ months = 5 years

$k = 12$ (monthly) Future value of A = ?

$$A = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{\left(1+\frac{i}{k}\right)^{nk}-1\right\}$$

$$A = \left(1+\frac{0.1}{12}\right)\left(\frac{1{,}600}{\frac{0.1}{12}}\right)\left\{\left(1+\frac{0.1}{12}\right)^{5\times 12}-1\right\}$$

$$A = \left(\frac{12.1}{12}\right)\times\left(\frac{1,600\times12}{0.1}\right)\left\{\left(\frac{12.1}{12}\right)^{60}-1\right\}$$

$A = \dfrac{12.1\times1,600}{0.1}$ $\{(1.0083)^{60} - 1\}$ Evaluation of $(1.0083)^{60}$:

$A = \dfrac{12.1\times1,600}{0.1}(1.622 - 1)$ $x = (1.0083)^{60}$

$A = 121 \times 1,600 \times 0.622$ $\log x = 60 \log (1.0083)$

$A = 1,20,419.2$ $= 60\ (0.0035) = 0.21$

$\therefore\ x = \text{antilog}\ (0.21) = 1.622$

He will receive Rs.1,20,419.2 on maturity.

[Note: Using calculator, value of $(1.00083)^{60}$= 1.642 and A = 1,24,291.2]

Ex.13 At the beginning of each month, Rs.200 is deposited in a saving account that pays 8% compounded monthly. Find the balance in the account at the end of 3 years.

Solution : Here, $a = 200$; $i = \dfrac{R}{100} = \dfrac{8}{100} = 0.08$; $n = 36$ months = 3 years

$k = 12$ (monthly) Future value of A = ?

$$A = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{\left(1+\frac{i}{k}\right)^{nk}-1\right\}$$

$$A = \left(1+\frac{0.08}{12}\right)\left(\frac{200}{\frac{0.08}{12}}\right)\left\{\left(1+\frac{0.08}{12}\right)^{3\times12}-1\right\}$$

$$A = \left(\frac{12.08}{12}\right)\times\left(\frac{200\times12}{0.08}\right)\left\{\left(\frac{12.08}{12}\right)^{36}-1\right\}$$

$A = \dfrac{12.08\times200}{0.08}\{(1.0066)^{36} - 1\}$ Evaluation of $(1.0066)^{36}$:

$A = 30,200\ (1.2612 - 1)$ $x = (1.0066)^{36}$

$A = 30,200 \times 0.2612$ $\log x = 36 \log (1.0066)$

$A = 7,888.24$ $= (36)\ (0.0028) = 0.1008$

$\therefore x = \text{antilog}\ (0.1008) = 1.2612$

Balance in the account at the end of 3 years Rs. 7,888.24 on maturity.

[Note: Using calculator value of $(1.0066)^{36}$= 1.2672 and A = 8,069.44]

Ex.14 What sum should Ms. Aarti deposit in the beginning of January, April, July and October each year at 16% rate of interest so that she would receive Rs.5,00,000 at the end of 12 years?

Solution : Here, A = 5,00,000; $i = \frac{R}{100} = \frac{16}{100} = 0.16$; $n = 12$;

$k = 4$ (quarterly); Annuity a = ?

$$A = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{\left(1+\frac{i}{k}\right)^{nk}-1\right\}$$

$$5,00,000 = \left(1+\frac{0.16}{4}\right)\left(\frac{a}{\frac{0.16}{4}}\right)\left\{\left(1+\frac{0.16}{4}\right)^{12\times4}-1\right\}$$

$$5,00,000 = (1 + 0.04)\left(\frac{a}{0.04}\right)\{(1 + 0.04)^{48} - 1\}$$

$$5,00,000 = 1.04 \times \frac{a}{0.04}\{(1.04)^{48} - 1\}$$

$$5,00,000 = 1.04 \times \frac{a}{0.04}(6.546 - 1)$$

$$5,00,000 = 1.04 \times \frac{a}{0.04} \times 5.546$$

$$a = \frac{5,00,000 \times 0.04}{1.04 \times 5.546}$$

$\therefore a = 3,467.5$

Evaluation of $(1.04)^{48}$:

$x = (1.04)^{48}$

$\log x = 48 \log (1.04)$

$= 48 (0.0170) = 0.8160$

$\therefore x = \text{antilog } (0.8160) = 6.546$

Ms. Arti should deposit Rs.3,467.5 in the beginning of every quarter.

[Note: Using calculator value of $(1.04)^{48}$= 6.5705 and A = 3,452.25]

:: EXERCISE – 1 ::

1. Mr. Rajesh deposits Rs.8000 with HDFC Bank at the end of every year at 13% rate of interest. What sum will he receive from HDFC Bank at the end of 12 years ? **[$(1.13)^{12}$= 4.3345]** ***(Ans.205200)***

2. Mr. Rahi deposited Rs.7500 at the end of every year since 2016 at 15.5% rate of interest with a bank. What sum was accumulated in his account after depositing the sum in the year 2020? **[$(1.155)^{5}$= 2.0554]** ***(Ans. 51068)***

3. Mr. Hira deposits Rs. 350 at the end of every month in his provident fund account. The rate of interest is 12%. What sum will be accumulated in his provident fund account at the end of 30 years of his job? **[$(1.01)^{360}$= 2.0554]** ***(Ans. 122323.6)***

4. Mr. Niraj deposits Rs. 2000 at 18% rate of interest at the end of every three months with a finance company. What amount will he receive from the finance company at the end of 10 years? **[$(1.045)^{40}$= 5.8163]** ***(Ans. 214062.22)***

5. Mr. Niraj's daughter Riya is 5 years old. Mr. Niraj deposits certain fixed sum at 12% rate of interest at the end of every month, so that Riya would receive Rs.4,00,000 when she is 25 years old. Find the sum deposited every month by Mr. Manoj. **[$(1.01)^{240}$= 10.8925]** ***(Ans. 405)***

6. Mr. Joseph opens a P.P.F. A/c. with Federal Bank for 15 years. In the beginning of each year, he deposits Rs.5000. If the rate of interest is 12%, find out what sum will he receive at the end of 15 years. **[$(1.12)^{15}$= 5.4736]** ***(Ans. 208768)***

7. In the beginning of each year, Mr. X deposits Rs.3000 in the recurring account. If the rate of interest is 10.5%, what sum will he receive at the end of 10 years? **[$(1.105)^{10}$= 2.714]** ***(Ans. 54113)***

8. Mrs. Sweta's daughter Soly is 12 years old. Mrs. Sweta deposits certain fixed sum at 15% rate of interest with VLS Finance in the beginning of every year so that she would receive Rs.5,00,000 for the wedding of Soly when she attains the age of 24 years. Find the sum deposited by Mrs. Sweta in the beginning of each year. **[$(1.15)^{12}$= 5.3502]** ***(Ans. 14992)***

9. What fixed sum should Mr. Gaurav deposit at 18% rate of interest in the beginning of every month so that at the end of 5 years' period of his recurring A/c., he will receive Rs.1,00,000? **[$(1.015)^{60}$= 2.4432]** ***(Ans. 1024)***

10. Mr. Bhuvan deposits certain fixed sum at 12% rate of interest in the beginning of every January and July with Federal Bank Ltd. for 12 years. Find the sum he deposited, if he receives Rs. 5,00,000 on maturity. **[$(1.06)^{24}$= 4.0489]** ***(Ans. 9283)***

11. Mr. Alex opens a recurring account for 12 years. In that account, he deposits Rs. 800 on 1st January, 1st may and 1st September every year. If the rate of interest is 15%, what sum will he receive on maturity? **[$(1.05)^{36}$= 5.7918]** ***(Ans. 80502)***

14.4 Present Value Of Immediate Annuity

- **Present Value of an Immediate Annuity Certain :**

We frequently come across problems of finding the present value or accumulated amount of an annuity. The present value of an annuity is simply the sum of the present values of each of its periodical payments as at the commencement of the annuity. Similarly, the amount or accumulated value of the annuity is simply the sum of the accumulated amounts of the periodical payments as at the end of the period of the annuity.

Let us find the present value of an immediate annuity of Re.1 p.a. payable for n years certain at rate of interest of i per unit per annum. This present value is denoted by the symbol P. Under this annuity, n payments are to be made, the first one at the end of the 1st year, the second one at the end of the 2nd year and so on, the last payment being made at the end of n years. To get the present value of the annuity, we have to find the present value of each of these n payments as at the commencement, and add them up.

As at the commencement of the annuity,

value of the 1st payment of 1 payable at the end of 1 year = $\frac{1}{(1+i)}$

value of the 2nd payment of 1 payable at the end of 2 years = $\frac{1}{(1+i)^2}$

value of the nth payment of 1 payable at the end of n years = $\frac{1}{(1+i)^n}$

Adding, we get the present value of the annuity

$$P = \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + ... + \frac{1}{(1+i)^n}.$$

The expression above is a geometric progression whose first term is $\frac{1}{(1+i)}$, common ratio is $\frac{1}{(1+i)}$ and number of terms is *n*.

Therefore applying the formula for the sum of a G.P., we obtain the equation for present value of the annuity as under:

$$P = \frac{1}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

- **Calculate the present value of annuity to be paid once in a year at the end of each period.**

Ex.15 Mr. Miraj purchased T.V. on 1-1-2024 by installment. At the end of every year, he has to be pay equal installments of Rs.6,400 for 8 years. If the rate of interest is 14%, then find the cash price of the T.V..

Solution : Here, a = 6,400; $i = \frac{R}{100} = \frac{14}{100}$ = 0.14,n = 8; Present value P = ?

$$P = \frac{a}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

$$P = \frac{6,400}{0.14}\left(1 - \frac{1}{(1+0.14)^8}\right)$$

$$P = \frac{6,400}{0.14}\left(1 - \frac{1}{(1.14)^8}\right)$$

$$P = \frac{6,400}{0.14}\left\{1 - \frac{1}{2.8525}\right\}$$

$$P = \frac{6,400}{0.14}\left\{\frac{2.8525 - 1}{2.8525}\right\}$$

$$P = \frac{6,400}{0.14} \times \frac{1.8525}{2.8525}$$

$$P = 29,688.25$$

Hence, the cash price of T.V. is Rs.29,688.25.

Ex.16 A person purchase on A.C. on 1st Jan. 2024, in installment at the end of every year. He has to pay equal installment of Rs. 5,000 for 8 years. If the rate of interest is 14% then find the cash price.

Solution : Here, a = 5,000; $i = \frac{R}{100} = \frac{14}{100}$ =0.14; n = 8; Present value P = ?

$$P = \frac{a}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

$$P = \frac{5,000}{0.14}\left(1 - \frac{1}{(1+0.14)^8}\right)$$

$$P = \frac{5,000}{0.14}\left(1 - \frac{1}{(1.14)^8}\right)$$

$$P = \frac{5,000}{0.14}\left\{1 - \frac{1}{2.8525}\right\}$$

$$P = \frac{5,000}{0.14}\left\{\frac{2.8525-1}{2.8525}\right\}$$

$$P = \frac{5,000}{0.14} \times \frac{1.8525}{2.8525}$$

$$P = 23,193.94$$

Hence, the cash price of the A.C. is Rs.23,193.94.

Ex.17 Viral purchased a machine on 1–1–2023. He agrees to pay 12 equal installments each of Rs.7500 to be paid at the end of every year inclusive of interest 12.5%. Find out its cash price.

Solution : Here, a = 7,500; $i = \frac{R}{100} = \frac{12.5}{100}$ =0.125; n = 12; Present value P = ?.

$$P = \frac{a}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

$$P = \frac{7,500}{0.125}\left(1 - \frac{1}{(1+0.125)^{12}}\right)$$

$$P = \frac{7{,}500}{0.125}\left(1 - \frac{1}{(1.125)^{12}}\right)$$

$$P = \frac{7{,}500}{0.125}\left\{1 - \frac{1}{4.1098}\right\}$$

$$P = \frac{7{,}500}{0.125}\left\{\frac{4.1098 - 1}{4.1098}\right\}$$

$$P = \frac{7{,}500}{0.125} \times \frac{3.1098}{4.1098}$$

P = 45,400,75

Hence, the cash price of the machine is Rs.45,400,75.

Ex.18 Mahek Trading Co. plans to buy a machine worth Rs.3,50,000 on 1-1-2024 whose expected life is 12 years. It is assumed that due to this machine the cash-flow per annum of the company will increase by Rs.45,000 deducting all the expenses and taxes. The rate of interest is 12.5%. Decide whether it is advisable or not to buy the machine.

Solution : Here, a = 45,000; n = 12; $i = \frac{R}{100} = \frac{12.5}{100}$ =0.125 and P = ?

$$P = \frac{a}{i}\left\{1 - \frac{1}{(1+i)^n}\right\}$$

$$P = \frac{45{,}000}{0.125}\left\{1 - \frac{1}{(1+0.125)^{12}}\right\}$$

$$P = 3{,}60{,}000\left\{1 - \frac{1}{(1.125)^{12}}\right\}$$

$$P = 3{,}60{,}000\left\{1 - \frac{1}{4.1098}\right\}$$

$$P = 3{,}60{,}000\left(\frac{4.1098 - 1}{4.1098}\right)$$

$$P = \frac{3{,}60{,}000 \times 3.1098}{4.1098}$$

P = 2,72,404.5

Thus, the present value of the additional cash flow available is Rs.2,72,404.5 while the purchase price of the machine is Rs.3,50,000.

Hence, it is not advisable to buy the machine.

Ex.19 Pepsi Food and Beverages borrowed loan of Rs.1,20,00,000 at 15% rate of interest from Bank of India on 1-1-2024. The loan is to be repaid by 20 equal installments, to be paid annually beginning from 31-12-2024. Find the sum of each installment.

Solution : Here, P = 1,20,00,000; n = 20; $i = \frac{R}{100} = \frac{15}{100}$ = 0.15; a = ?

$$P = \frac{a}{i}\left\{1-\frac{1}{(1+i)^n}\right\}$$

$$1{,}20{,}00{,}000 = \frac{a}{0.15}\left\{1-\frac{1}{(1+0.15)^{20}}\right\}$$

$$1{,}20{,}00{,}000 \times 0.15 = a\left\{1-\frac{1}{(1.15)^{20}}\right\}$$

$$18{,}00{,}000 = a\left\{\frac{16.37-1}{16.37}\right\}$$

$$18{,}00{,}000 = a\times\frac{15.37}{16.37}$$

$$a = \frac{18{,}00{,}000\times16.37}{15.37}$$

a =19,17,111.25

Evaluation of $(1.15)^{20}$:

$x = (1.15)^{20}$

$\log x = 20 \log (1.15)$

$= 20\ (0.0607) = 1.214$

$\therefore x$ = antilog (1.214) = 16.37

Hence, the sum to be paid in each installment is Rs.19,17,111.25

Ex.20 Reliance industries limited took loan of Rs.15,00,000. On 1-1-2024 at 12% rate of compound interest from State Bank of India to purchase a plant of vatva industrial estate, Ahmadabad. Repayment starts from 31-12-2024. If the number of yearly instilments is 10, each of equal amount and inclusive of interest, then find out sum of an installment.

Solution : Here, P = 15,00,000;n = 10; $i = \frac{R}{100} = \frac{12}{100}$ = 0.12; a = ?

$$P = \frac{a}{i}\left\{1-\frac{1}{(1+i)^n}\right\}$$

$$15{,}00{,}000 = \frac{a}{0.12}\left\{1-\frac{1}{(1+0.12)^{10}}\right\}$$

$$15{,}00{,}000 \times 0.12 = a\left\{1-\frac{1}{(1.12)^{10}}\right\}$$

$$1,80,000 = a\left\{\frac{3.1058-1}{3.1058}\right\}$$

$$1,80,000 = a \times \frac{2.1058}{3.1058}$$

$$a = \frac{1,80,000 \times 3.1058}{2.1058}$$

$$a = 2,65,478.20$$

Hence, the sum to be paid in each installment is Rs. 2,65,478.20

- **Calculate the present value of annuity to be paid more than once in a year at the end of each period.**

Ex.21 Mr. Ramesh purchased Car on 1-1-2024 by installments system. He paid Rs.20,000 cash at the time of purchase and has to pay Rs.20,000 at the end of every month for 3 years. If the rate of interest is 12 %, then find the cash price of the car.

Solution : Here, $a = 20,000$; $i = \frac{R}{100} = \frac{12}{100} = 0.12$; $k = 12$(monthly);

$n = 3$ Present value of P = ?

$$P = \frac{a}{\frac{i}{k}}\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$P = \frac{20,000}{\frac{0.12}{12}}\left\{1-\frac{1}{\left(1+\frac{0.12}{12}\right)^{3\times12}}\right\}$$

$$P = \frac{20,000}{0.01}\left\{1-\frac{1}{(1+0.01)^{36}}\right\}$$

$$P = 20,00,000\left\{1-\frac{1}{(1.01)^{36}}\right\}$$

Evaluation of $(1.01)^{36}$:

$$P = 20,00,000\left\{1-\frac{1}{1.429}\right\}$$

$x = (1.01)^{36}$

$$P = 20,00,000\left\{\frac{1.429-1}{1.429}\right\}$$

$\log x = 36 \log (1.01)$

$$P = 20,00,000 \times \frac{0.429}{1.429}$$

$= 36\ (0.0043) = 0.1548$

P = 6,00,420 $\therefore x$ = antilog (0.1548) = 1.429

Hence, the present value of annuity is Rs.6,00,420.

Now, cash price of the car = down payment + present value of annuity
= 20,000 + 6,00,420

= 6,20,420

[Note: Using calculator value of $(1.01)^{36}$= 1.4308 and A = 6,02,000]

Ex.22 Mr. Shah purchased an office worth Rs.14,50,000 on 1-4-2024. He paid Rs.2,50,000 at the time of purchase and the remaining sum is to be paid by equal installments inclusive of interest every quarter for next 5 year. If the rate of interest is 16%, then find the sum to be paid in each quarterly installment.

Solution : Sum to be paid by installments = value of office – down payment

= 14,50,000 – 2,50,000

= 12,00,000

Here, P = 12,00,000; n = 5; $i = \frac{R}{100} = \frac{16}{100}$ = 0.16; k = 4(quarterly); a = ?

$$P = \frac{a}{\frac{i}{k}}\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$12,00,000 = \frac{a}{\frac{0.16}{4}}\left\{1-\frac{1}{\left(1+\frac{0.16}{4}\right)^{5\times4}}\right\}$$

$$12,00,000 = \frac{a}{0.04}\left\{1-\frac{1}{(1+0.04)^{20}}\right\}$$

$$12,00,000 \times 0.04 = a\left\{1-\frac{1}{(1.04)^{20}}\right\}$$

Evaluation of $(1.04)^{20}$:

$x = (1.04)^{20}$

$\log x = 20 \log (1.04)$

$= 20\ (0.0170)$

$= 0.3400$

$$48,000 = a\left\{1-\frac{1}{2.188}\right\}$$

$$48,000 = a\left\{\frac{2.188-1}{2.188}\right\}$$

$$48,000 = a \times \frac{1.188}{2.188}$$

$$a = \frac{48{,}000 \times 2.188}{1.188} \qquad \therefore x = \text{antilog}(0.34) = 2.188$$

a = 88,404

Hence, the sum to be paid in each quarterly installment is Rs.88,404.

[Note: Using calculator value of $(1.04)^{20}$= 2.1911 and A = 88,298.88]

Ex.23 Calculate the amount of each monthly installment. If loan of Rs.10,000 is to be paid in 30 equal monthly installments, when rate of compound interest is 4% per annum.

Solution: Here, P = 10,000; nk = 30; $i = \frac{R}{100} = \frac{4}{100}$ = 0.04; k = 12; a = ?

$$P = \frac{a}{\frac{i}{k}}\left\{1 - \frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$10{,}000 = \frac{a}{\frac{0.04}{12}}\left(1 - \frac{1}{\left(1+\frac{0.04}{12}\right)^{30}}\right)$$

$$10{,}000 = \frac{a}{0.0033}\left\{1 - \frac{1}{(1+0.0033)^{30}}\right\}$$

$$10{,}000 \times 0.0033 = a\left\{1 - \frac{1}{(1.0033)^{30}}\right\} \qquad \text{Evaluation of } (1.0033)^{30}:$$

$$33 = a\left\{1 - \frac{1}{1.10388}\right\} \qquad x = (1.0033)^{30}$$

$$33 = a\left\{\frac{1.10388 - 1}{1.10388}\right\} \qquad \log x = 30 \log (1.0033)$$

$$33 = a \times \frac{0.10388}{1.10388} \qquad = 30\,(0.0014) = 0.042$$

$$a = \frac{33 \times 1.10388}{0.10388} \qquad \therefore x = \text{antilog}(0.042) = 1.10388$$

$\therefore a$ = 350.68

Hence, the sum to be paid in each quarterly installment is Rs.350.68.

14.5 Present Value Of Annuity Due

Annuity is payable or receivable annually at the beginning of each year.

$$P = (1+i)\ \frac{a}{i}\left\{1-\frac{1}{(1+i)^n}\right\}$$

➢ **The present value of annuity to be paid annuity at the beginning of each year.**

Ex.24 Mr. Suraj purchased a car by installment system. Right from the date of purchase, he has to pay Rs.1,60,000 annually for 10 years. If the rate of interest is 12.5%, find the cash price of the car.

Solution: Here, a = 1,60,000; $i = \frac{R}{100} = \frac{12.5}{100} = 0.125$; $n = 10$; P = ?

$$P = (1+i)\left(\frac{a}{i}\right)\left\{1-\frac{1}{(1+i)^n}\right\}$$

$$P = (1+0.125)\left(\frac{1,60,000}{0.125}\right)\left\{1-\frac{1}{(1+0.125)^{10}}\right\}$$

$$P = 14,40,000\ \left\{1-\frac{1}{3.2473}\right\}$$

$$P = 14,40,000\ \left\{\frac{3.2473-1}{3.2473}\right\}$$

$$P = 14,40,000 \times \frac{2.2473}{3.2473}$$

$$P = 9,96,554.68$$

Hence, the cash price of the car is Rs.9,96,554.68.

Ex.25 Mr. Jayesh purchased a Car worth Rs.14,00,000 by installment system. 10 annual installments are to be paid, the first of which is to be paid on the date of purchase. If the rate of interest is 12%, then find the sum to be paid in each annual installment.

Solution : Here, P = 14,00,000; $i = \frac{R}{100} = \frac{12}{100} = 0.12$; $n = 10$; Annuity a = ?

$$P = (1+i)\frac{a}{i}\left\{1-\frac{1}{(1+i)^n}\right\}$$

$$14,00,000 = (1+0.12)\ \frac{a}{0.12}\left\{1-\frac{1}{(1+0.12)^{10}}\right\}$$

$$\frac{14,00,000 \times 0.12}{1.12} = a\left\{1-\frac{1}{3.105}\right\}$$

$$1,50,000 = a\left\{\frac{3.105-1}{3.105}\right\}$$

$$1,50,000 = a \times \frac{2.105}{3.105}$$

$$a = 2,21,258.9$$

Hence, the sum to be paid in each annual installment is Rs.2,21,259.

- **The present value of annuity to be paid more than once in a year at the beginning of each period.**

  Annuity is payable or receivable more than once in a year at the beginning of each period:

$$P = \left(1+\frac{i}{k}\right)\frac{a}{\frac{i}{k}}\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

Ex.26 Mr. Paresh borrowed a certain sum on 1-1-2023 at 16% rate of interest. The loan is to be repaid by installments of Rs.4000, each to be paid in the beginning of January, April, July and October every year for 8 years. Find the sum he borrowed.

Solution : Here, $a = 4{,}000$; $i = \frac{R}{100} = \frac{16}{100} = 0.16$; $n = 8$;

$k = 4$ (Quarterly); $P = ?$

$$P = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$P = \left(1+\frac{0.16}{4}\right)\left(\frac{4{,}000}{\frac{0.16}{4}}\right)\left\{1-\frac{1}{\left(1+\frac{0.16}{4}\right)^{8\times4}}\right\}$$

$$P = (1+0.04)\left(\frac{4{,}000}{0.04}\right)\left\{1-\frac{1}{(1+0.04)^{32}}\right\}$$

$$P = (1.04)(1{,}00{,}000)\left\{1-\frac{1}{(1.04)^{32}}\right\}$$

Evaluation of $(1.04)^{32}$:

$$P = 1{,}04{,}000\left\{1-\frac{1}{3.499}\right\}$$

$x = (1.04)^{32}$

$$P = 1{,}04{,}000\left\{\frac{3.499-1}{3.499}\right\}$$

$\log x = 32 \log (1.04)$

$$P = 1{,}04{,}000 \times \frac{2.499}{3.499}$$

$= 32\ (0.0170) = 0.544$

$P = 74{,}277.22$

$\therefore x = \text{antilog}\ (0.544) = 3.499$

Hence, the sum borrowed by Mr. Paresh is Rs.74,277.22.

[Note: Using calculator value of $(1.04)^{32}$= 3.508 and A = 74,353.47]

Ex.27 Mr. Stephen purchased a Car worth Rs.5,55,000 by installment system.

The sum with interest is to be paid by 20 half yearly installments, the first of which is to be paid on the date of purchase. If the rate of interest is 20%, then find the sum to be paid in each half yearly installment.

Solution: Here, P = 5,55,000; n = 20 half yearly installments = 10 years;

$$i = \frac{R}{100} = \frac{20}{100} = 0.20; k = 2 \text{ (half yearly)}; \text{ Annuity } a = ?$$

$$P = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$5{,}55{,}000 = \left(1+\frac{0.20}{2}\right)\left(\frac{a}{\frac{0.20}{2}}\right)\left\{1-\frac{1}{\left(1+\frac{0.20}{2}\right)^{10\times 2}}\right\}$$

$$5{,}55{,}000 = \left(1+0.10\right)\left(\frac{a}{0.10}\right)\left\{1-\frac{1}{\left(1+0.10\right)^{20}}\right\}$$

$$5{,}55{,}000 = \left(1.10\right)\frac{a}{0.10}\left\{1-\frac{1}{\left(1.10\right)^{20}}\right\}$$

Evaluation of $(1.10)^{20}$:

$x = (1.10)^{20}$

$\log x = 20 \log (1.10)$

$= 20 (0.0413) = 0.826$

$\therefore x = \text{antilog } (0.826) = 6.730$

$$\frac{5{,}55{,}000\times 0.10}{1.10} = a\left\{1-\frac{1}{6.730}\right\}$$

$$50{,}454.55 = a\left\{\frac{6.730-1}{6.730}\right\}$$

$$50{,}454.55 = a\times\frac{5.730}{6.730}$$

$$a = \frac{50{,}454.55 \times 6.730}{5.730}$$

a = 59,259.23

Hence, sum to be paid in each half yearly installment is Rs. 59,259.23.

Ex.28 Mr. X purchased a house cost Rs.8,00,000 on the basis of 24 monthly installment. Starting from purchased date 24 installment are paid with interest. If the rate of interest 18% per annum, then find the amount of his monthly installment.

Solution : Here, P = 8,00,000; n = 24 monthly installments = 2 years;

$$i = \frac{R}{100} = \frac{18}{100} = 0.18; k = 12; \text{ Annuity } a = ?$$

$$P = \left(1+\frac{i}{k}\right)\left(\frac{a}{\frac{i}{k}}\right)\left\{1-\frac{1}{\left(1+\frac{i}{k}\right)^{nk}}\right\}$$

$$8,00,000 = \left(1+\frac{0.18}{12}\right)\left(\frac{a}{\frac{0.18}{12}}\right)\left\{1-\frac{1}{\left(1+\frac{0.18}{12}\right)^{12\times 2}}\right\}$$

$$8,00,000 = (1+0.015)\left(\frac{a}{0.015}\right)\left\{1-\frac{1}{(1+0.015)^{24}}\right\}$$

$$8,00,000 = (1.015)\frac{a}{0.015}\left\{1-\frac{1}{(1.015)^{24}}\right\}$$

Evaluation of $(1.015)^{24}$:

$x = (1.015)^{24} = 6.730$

$\log x = 24 \log (1.015)$

$= 24\ (0.0064) = 0.1536$

$\therefore x = \text{antilog}\ (0.1536) = 1.4295$

$$\frac{8,00,000\times 0.015}{1.015} = a\left\{1-\frac{1}{1.4295}\right\}$$

$$11,822.66 = a\left\{\frac{1.4295-1}{1.4295}\right\}$$

$$11,822.66 = a\times\frac{0.4295}{1.4295}$$

$$a = \frac{11,822.66 \times 1.4295}{0.4295}$$

$$a = 39,349.22$$

Hence, sum to be paid in each half yearly installment is Rs.39,349.22.

14.6 APPLICATIONS

[1] SINKING FUND :

Sometimes, companies create a fund to meet some predetermined debts or liabilities or the expense of new assets out of their profit at the end of every accounting year. This fund is known as Sinking Fund or Pay Back Fund.

It is the fund credited for a specified purpose by way of sequence of periodic payments over a time period at a specified interest rate. Interest is compounded at the end of every period. Size of the sinking fund deposit is computed from

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

[2] Capital Expenditure (investment decision):

Capital expenditure means purchasing an asset (which results in outflows of money) today in anticipation of benefits (cash inflow) which would flow across the life of the investment. For taking investment decision we compare the present value of cash outflow and present value of cash inflows. If present value of cash inflows is greater than present value of cash outflows, decision should be in the favor of investment. Let us see how we take capital expenditure (investment) decision by some example.

Ex.29 Venus Ltd. has issued 2000 debentures each of Rs.100. The sum of the debentures is to be paid back after 7 years. To meet the said liability, it has been decided to create a sinking fund and invest it at 14.5% rate of interest at the end of each year. Find the sum to be transferred to sinking fund every year.

Solution : Here, $A = 100 \times 2000$ debentures = Rs.2,00,000.

$$i = \frac{R}{100} = \frac{14.5}{100} = 0.145; \; n = 7$$

The sum to be transferred to sinking fund a = ?

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

$$2{,}00{,}000 = \frac{a}{0.145}\left\{(1+0.145)^7 - 1\right\}$$

$$2{,}00{,}000 \times 0.145 = a\left\{(1.145)^7 - 1\right\}$$

$$29{,}000 = a\,(2.580 - 1)$$

$$29{,}000 = a \times 1.580$$

$$a = \frac{29{,}000}{1.580} = 18{,}354.43$$

Hence, every year Rs. 18,354.43 should be transferred to sinking fund.

Ex.30 Adani Ltd. issued 20,000 debentures of Rs.100 each to be redeemed after 10 years. In order to get a fund to redeem these debentures, it was decided to create a sinking fund and to invest it at 12% rate of interest. Find out the sum to be transferred to this sinking fund every year.

Solution : Here, $A = 100 \times 20{,}000$ debentures = Rs.20,00,000.

$$i = \frac{R}{100} = \frac{12}{100} = 0.12; \; n = 10$$

The sum to be transferred to sinking fund a = ?

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

$$20{,}00{,}000 = \frac{a}{0.12}\left\{(1+0.12)^{10} - 1\right\}$$

$$20{,}00{,}000 \times 0.12 = a\left\{(1.12)^{10} - 1\right\}$$

$2,40,000 = a\,(3.1058 - 1)$

$2,40,000 = a \times 2.1058$

$$a = \frac{2,40,000}{2.1058} = 1,13,970.94$$

Every year Rs.1,13,970.94 should be transferred to sinking fund.

Ex.31 Shree Travels Ltd. Purchased Volvo Bus worth Rs.85,00,000. Its expected life is 9 years. It is estimated that after 9 years, the price of the bus will increase by 50%. To buy a new Volvo Bus, it has been decided to create a sinking fund and invest it at 14.5% rate of interest. Find the sum to be transferred to sinking fund every year.

Solution: The sum required to buy a new Volvo Bus after 9 years

= current price 85,00,000 + 50% price rise

= 85,00,000+ 42,50,000 = 1,27,50,000

Here, A = 1,27,50,000; n = 9; $i = \frac{R}{100} = \frac{14.5}{100}$ = 0.145; a = ?

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

$$1,27,50,000 = \frac{a}{0.145}\left\{(1+0.145)^9 - 1\right\}$$

$$1,27,50,000 \times 0.145 = a\left\{(1.145)^9 - 1\right\}$$

$18,48,750 = a\,(3.383 - 1)$

$18,48,750 = a \times 2.383$

$$A = \frac{18,48,750}{2.383} = 7,75,808$$

Hence, Rs.7,75,808 should be transferred to sinking fund every year.

Ex.32 Sweta Limited purchased a machine for Rs.2,50,000. The expected life of it is 10 years. It is assumed that after 10 years a new machine would cost them 40% more. Under this assumption the company has decided to create a sinking fund and invest them at 15% rate of compound interest. Find out the sum to be transferred every year to this fund.

Solution: The sum required to buy a new machine after 10 years

= current price 2,50,000 + 40% price rise

= 2,50,000 + 1,00,000 = 3,50,000

Here, A = 3,50,000; n = 10; $i = \frac{R}{100} = \frac{15}{100}$ = 0.15; a = ?

$$A = \frac{a}{i}\left\{(1+i)^n - 1\right\}$$

$$3,50,000 = \frac{a}{0.15}\left\{(1+0.15)^{10} - 1\right\}$$

$$3,50,000 \times 0.15 = a\left\{(1.15)^{10} - 1\right\}$$

$$52,500 = a\,(4.0455 - 1)$$

$$52,500 = a \times 3.0455$$

$$a = \frac{52,500}{3.0455} = 17,238.55$$

Hence, Rs.17,238.55 should be transferred to sinking fund every year.

:: EXERCISE – 2 ::

1. Mr. Patel purchased a computer on 1-4-2023. He has to pay Rs.15,000 every year on 31st March for 10 years. If the rate of interest is 15%, then find the cash price of the computer.

$[(1.15)^{10} = 4.0455]$

(Ans.75282)

2. Mr. Hiren buys Air Conditioner by installments. He has to pay Rs.6,000 at the end of each year for 5 years. If the rate of interest is 12.5%, then find the cash price of the Air Conditioner. **$[(1.125)^{5} = 1.802]$** ***(Ans.21363)***

3. Mr. Miraj purchases a scooter on 1-1-2024 by installments. He pays Rs.4000 cash on the purchase and has to pay Rs.600 at the end of every month for 5 years. If the rate of interest is 18%, find the cash price of the scooter. **$[(1.015)^{60} = 2.4432]$**

(Ans.27628)

4. Mr. Naresh purchased Car on 1-1-2023 by installments. He paid Rs.20,000 cash on buying and has to pay Rs.3,000 on every 30th June and 31st December for 6 years. If the rate of interest is 14%, find the cash price of the machine. **$[(1.07)^{12} = 2.2522]$** ***(Ans.43830)***

5. Mr. Nihar purchased a tempo-track by installment system. Right from the date of purchase, he has to pay Rs. 20,000 annually for 5 years. If the rate of interest is 10%, then find the cash price of the tempo-track. **$[(1.1)^{5} = 1.6105]$** ***(Ans. 83396)***

6. Chirag purchased a fax machine by installment system. Right from the date of purchase, it has to pay Rs. 920 annually for 18 years. If the rate of interest is 11.5 %, find the cash price of the fax machine. **$[(1.115)^{18} = 7.0949]$** ***(Ans.7663)***

7. Mr. Bin borrowed a certain sum from Bank of India at 12% rate of interest on 1-1-2024. The loan is to be repaid with interest by installment of Rs.1,200 cash, to be paid on 1st of January, April, July and October every year, for 8 years. Find the sum borrowed by Mr. Bin. **$[(1.03)^{32} = 2.575]$** ***(Ans.25200)***

8. Mr. Nath borrowed a certain sum from a financier on 1-1-2024 at 18% rate of interest. The loan is to be repaid with interest by paying installments of Rs. 36,000 cash on 1st January and 1st July every year, up to 6 years. Find the sum borrowed by Mr. Nath. **$[(1.09)^{12} = 2.8127]$** ***(Ans.280989)***

9. Mr. Parry borrowed Rs.7,50,000 at 16.5% rate of interest from a bank on 1-1-2024.The loan is to be repaid by 10 equal annual installments, inclusive of interest, beginning from 31-12-2024. Find the sum to be paid in each installment.
[$(1.165)^{10}$= 4.6053] ***(Ans.158100)***

10. Mr. Jani borrowed a loan of Rs.25,00,000 from ICICI Bank on 1-4-2020 at 14% rate of interest. The loan is to be repaid by 15 equal annual installments, inclusive of interest, of which the first installment is to be paid on 31-3-2021. Find the sum to be paid in each installment. **[$(1.14)^{15}$= 7.1379]** ***(Ans.407022)***

11. Mr. Paresh purchased cycle worth Rs.30,000 by installment system on 1-1-2024. He paid Rs.5,000 cash on the date of purchase and the remaining amount with interest at 12% rate is to be paid by 24 equal monthly installments, to be paid at the end of each month. Find the sum of each monthly installment. **[$(1.01)^{24}$ = 1.2697]**
(Ans.1180)

12. Mr. Shah purchased a car worth Rs.7,50,000 by installment system on 1-1-2024. He paid Rs.1,50,000 cash on the date of purchase. The remaining sum with interest is to be paid by installment payable on 30th June and 31st December for 4 years. If the rate of interest is 18%, then find the sum to be paid in each half yearly installment. **[$(1.09)^{8}$= 1.9925]** ***(Ans.108400)***

13. Mr. Maulik purchased a second-hand car worth Rs.99,000 by installments. The sum with interest is to be paid by 15 equal annual instilments, the first of which is to be paid on the date of purchase. If the rate of interest is 10%, then find the sum of each installment. **[$(1.1)^{15}$= 4.1772]** ***(Ans.11833)***

14. Mr. Rahi purchased an office worth Rs. 4,20,000 by installment. 12 equal annual installments inclusive of interest, is to be paid on the date of purchase. If the rate of interest is 12%, find the sum to be paid in each installment. **[$(1.12)^{12}$= 3.8959]**
(Ans.60540)

15. Mr. Haresh borrowed Rs.2,20,000 at 12% rate of interest from Bank of India on 1-1-2024. The loan is to be repaid in 10 years by monthly equal installment, the first installment is to be paid on 1-1-2025. Find the sum to be paid in each monthly installment. **[$(1.01)^{120}$= 3.3003]** ***(Ans.3125)***

16. Explain : **[1]**Annuity **[2]**Sinking Fund

17. Explain the term annuity and describe its types.

➢ **Select the appropriate answer from the given alternative answer. (M.C.Q.)**

1. The present value of an annuity of Rs. 3000 for 15 years at 4.5% p.a.is
(a) Rs. 23809.41 (b)Rs. 32218.63 (c)Rs. 32908.41 (d)none of these

2. The amount of an annuity certain of Rs.150 for 12 years at 3.5% p.a.is
(a) Rs. 2190.28 (b)Rs. 1290.28 (c)Rs. 2180.28 (d)none of these

3. A loan of Rs.10,000 is to be paid back in 30 equal installments. The amount of each installment to cover the principal and at 4% p.a.is
(a) Rs. 587.87 (b) Rs. 587 (c)Rs. 578.87 (d)none of these

4. a = Rs. 1200, n = 12, i = 0.08, P = ?
Using the formula, $P = \frac{a}{i}\left[1 - \frac{1}{(1+i)^n}\right]$ value of P will be

(a) Rs. 3039 (b)Rs. 3990 (c)Rs. 9930 (d) none of these

5. a = Rs. 100, n = 10, i = 5%, the Future Value of annuity is

(a) Rs. 1258 (b)Rs. 2581 (c)Rs. 1528 (d)none of these

6. If the amount of an annuity after 25 years at 5% p.a.is Rs. 50,000, the annuity will be

(a) Rs. 1406.90 (b)Rs. 1046.90 (c)Rs. 1146.90 (d)none of these

7. A person invests Rs. 500 at the end of each year with a bank which pays interest at 10% p.a. annually. The amount standing to his credit one year after he has made his yearly investment for the 12^{th} time is.

(a) Rs. 11764.50 (b)Rs. 10000 (c)Rs. 12000 (d)none of these

8. The present value of annuity of Rs. 5,000 per annum for 12 years at 4% p.a. annually is

(a) Rs. 46000 (b)Rs. 46850 (c)Rs.15000 (d)none of these

9. The present value of an annuity of Rs.80 per annum for 20 years at 5% p.a. is

(a) Rs. 997 (appx.) (b)Rs. 900 (c)Rs. 1000 (d)none of these

10. A person bought a house, paying Rs.20,000 cash and Rs.4,000 at the end of each year for 25 years at 5% p.a. The cash price is

(a) Rs. 75000 (b)Rs. 76000 (c)Rs. 76392 (d)none of these

:: ANS ::

1. (b) **2.** (a) **3.** (c) **4.** (d) **5.** (a) **6.** (b) **7.** (a) **8.** (d)

9. (a) **10.** (c)

NOTES

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સ્વાધ્યાય: પરમં તપ:
સ્વાધ્યાય: પરમં તપ:
સ્વાધ્યાય: પરમં તપ:

શિક્ષણ, સંસ્કૃતિ, સદ્ભાવ, દિવ્યબોધનું ધામ
ડૉ. બાબાસાહેબ આંબેડકર ઓપન યુનિવર્સિટી નામ;
સૌને સૌની પાંખ મળે, ને સૌને સૌનું આભ,
દશે દિશામાં સ્મિત વહે હો દશે દિશે શુભ-લાભ.

અભણ રહી અજ્ઞાનના શાને, અંધકારને પીવો ?
કહે બુદ્ધ આંબેડકર કહે, તું થા તારો દીવો;
શારદીય અજવાળા પહોંચ્યાં ગુર્જર ગામે ગામ
ધ્રુવ તારકની જેમ ઝળહળે એકલવ્યની શાન.

સરસ્વતીના મયૂર તમારે ફળિયે આવી ગહેકે
અંધકારને હડસેલીને ઉજાસના ફૂલ મહેંકે;
બંધન નહીં કો સ્થાન સમયના જવું ન ઘરથી દૂર
ઘર આવી મા હરે શારદા દૈન્ય તિમિરના પૂર.

સંસ્કારોની સુગંધ મહેંકે, મન મંદિરને ધામે
સુખની ટપાલ પહોંચે સૌને પોતાને સરનામે;
સમાજ કેરે દરિયે હાંકી શિક્ષણ કેરું વહાણ,
આવો કરીયે આપણ સૌ
ભવ્ય રાષ્ટ્ર નિર્માણ...
દિવ્ય રાષ્ટ્ર નિર્માણ...
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