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Computer Oriented Numerical Methods

Dr. Babasaheb Ambedkar Open University

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Computer Oriented Numerical Methods

BLOCK-1: COMPUTER ARITHMETIC AND SOLVING NON-LINEAR EQUATIONS

BLOCK-2: SOLVING SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS AND INTERPOLATION

BLOCK-3: CURVE FITTING

BLOCK-4: NUMERICAL DIFFERENTIATION AND INTEGRATION

Computer Oriented Numerical Method

BLOCK1: Computer Arithmetic and Solving Non-Linear Equations

BLOCK 1: COMPUTER ARITHMETIC AND SOLVING NON-LINEAR EQUATIONS

Block Introduction

Science and Engineering often require numerical results involving all types of numbers, its arithmetic.

Numbers and numerical arithmetic or computation are the subjects of algebra, a branch of Mathematics. There are four basic arithmetic operations: addition, subtraction, multiplication, and division. Computer arithmetic involves the representation of integers and real numbers in computer systems, as well as the manipulation of those numbers through hardware circuits or software programs.

In this block, we will introduce different types of the number presentation of numbers, arithmetic operations on it. Concept of errors and numerical methods to solve the non-linear equations.

Unit:2 focuses on floating-point presentation of numbers and due to limitation of the systems errors are introduced in the numerical calculations, are also discussed here.

In Unit: 3, different types of equations and some numerical methods are discussed to obtain the solution/s of the non-linear equations.

Block Objective

This block aims to make students aware about arithmetic operations in computer systems, different presentations of the numbers, storage of real numbers and errors in the numerical calculation.

Finally, the block will clear the concept of the roots/solutions of an equation, graphical meaning of the root/solution of equations, different types of equations and various iterative methods to solve non-linear equations.

Block Structure

BLOCK1:

UNIT1 **Computer Arithmetic**

Objectives, Mathematical Background, Number presentation, Floating point Arithmetic, Let us Sum Up

UNIT2 **Errors**

Objectives, Errors, Different type's errors. Let us Sum Up

UNIT3 **Solving Non-Linear Equations**

Objectives, Introduction, Types of non-linear equations, Methods of solving non-linear equations, Convergence of iterative methods, let us Sum Up

UNIT 1 COMPUTER ARITHMETIC: INTRODUCTION

Unit Structure

1.0 LEARNING OBJECTIVES:

After learning this unit, you should be able to:

- Understand the floating-point arithmetic
- Understand different arithmetic operations on floating-point numbers
- Learn errors due to arithmetic operations

1.1 NUMBER SYSTEMS:

A number system is defined as a system of writing to express numbers. It is the mathematical notation for representing numbers of a given set by using digits or other symbols in a consistent manner. It provides a unique representation of every number and represents the arithmetic and algebraic structure of the figures. It also allows us to operate arithmetic operations like addition, subtraction and division.

The value of any digit in a number can be determined by:

- The digit
- Its position in the number
- The base of the number system

1.2 NUMBER REPRESENTATION:

There are many potential sources of error in mathematical computations. To try to acquire an application of how errors arise, it is first useful to observe how number are represented in computers.

Now, let us discuss how many different categories are there for different types of Data-types in details:

1.2.1 Floating point arithmetic

Fractional Quantities are generally represented in computers by floating point form. In floating point form a number is represented in three parts:

- One for Sign
- One for Fractional part
- One for Exponent

Thus, the representation of a float is of the following general form:

Float + sign * mantissa * $b^{exponent}$

For example: The Numbers (i) $0.7292 * 10^4$
(ii) $0.1516 * 10^{-13}$

Also written as (i) 0.7292 E 04
(ii) 0.1516 E -13

A floating-point number is said to be normalised if the first digit of the mantissa is always 1. The shifting of mantissa to the left till it's most significant digit is non-zero is known as normalisation. Zero is a special case, because its fractional part has all zeros and a zero exponent, so zero can never be normalised.

1.2.2 Addition operation

Addition operation is performed on normalized floating–point number if the exponents of the numbers is equal. If the exponents are not equal the exponent of the number with smaller exponent is made equal to larger exponent and its mantissa is modified.

For example, if 0.7253E2 and 0.5467E5 are to be added, then the decimal point of mantissa of 0.7253E2 is shifted by 3 positions (i.e. 5-2) to left hand side and the exponent is increased by 3. Due to which exponent of both numbers become equal and addition operation can be operated on them by adding their mantissas.

Example 1.1

Add 0.8475E6 to 0.4376E3

Solution:

The decimal point of the mantissa of 0.4376E3 is shifted three position to left. It becomes 0.0004, whereas the digits 6, 7, and 3 are chopped off i.e. truncated. The exponent is incremented by 3. Therefore, the number after normalization becomes 0.0004E6. These numbers can be added now as shown below:

Addend	0.0004E6
Augend	0.8475E6
Sum	0.8479E6

Example 1.2

Add 0.8475E2 to 0.4376E2

Solution:

Since The exponents are already equal, they can be directly added as:

Addend	0.4376E2
Augend	0.8475E2
Sum	1.2851E2

In this case, the mantissa of the sum is greater than 1.0, so the decimal point is shifted to the left by one position and the exponent is increased by 1. The last digit of mantissa of sum is truncated giving the normalized sum as 0.1285E3.

Example 1.3

Add 0.8475E99 to 0.4376E99

Solution:

Since the exponents are already equal, they can be directly added as:

Addend	0.4376E99
Augend	0.8475E99
Sum	1.2851E99

In this case, the mantissa of the sum is greater than 1.0, so the decimal point is shifted to the left by one position and the exponent is increased by 1. The last digit of the mantissa of sum is chopped off, giving the normalized sum as 0.1285E100. Since this number is greater than the largest number which our hypothetical computer can handle, it is a case of overflow and the system will indicate this condition.

1.2.3 Subtraction operation

Like addition operation, the Subtraction operation is performed on normalized floating–point number if the exponents of the numbers is equal. If the exponents are not equal the exponent of the number with smaller exponent is made equal to larger exponent and its mantissa is modified.

During normalization if exponent of difference becomes less than -99 than the system will show an error because the number is smaller than the smallest number which our hypothetical computers can store. And this situation is known as **underflow**.

For example, if 0.7253E2 and 0.5467E5 are to be subtracted, then the decimal point of mantissa of 0.7253E2 is shifted by 3 positions (i.e. 5-2) to left hand side and the exponent is increased by 3. Due to which exponent of both numbers become equal and subtraction operation can be operated on them by subtracting their mantissas.

Example 1.4

Subtract 0.8475E3 to 0.4376E6

Solution:

The number 0.8475E3 is normalized, thus resulting in number 0.0008E6. Now the number 0.0008E6 an be subtracted from 0.4376E6 as:

Minuend	0.4376E6
Subtrahend	0.0008E6
Difference	0.4368E6

Example 1.5

Subtract 0.8475E6 to 0.8476E6

Solution:

Since the exponents are already equal, the number 0.8475E6 can be directly subtracted from 0.8476E6 as:

Minuend	0.8476E6
Subtrahend	0.8475E6

Difference   0.0001E6

In this case, the mantissa of the difference is less than 0.1, so the decimal point is shifted three positions to the right and the exponent is decreased by 3. The resultant difference becomes 0.1E3.

Check Your Progress-1

1. Add $x_1 = 264.9$ to $x_2 = 26.00$

[A] 0.2909×10^3 [B] 0.2654×10^3
[C] 0.6754×10^{-3} [D] 0.7459×10^{-3}

2. Add 1.37 and 0.0269

[A] 0.139×10^1 [B] 0.153×10^2
[C] 0.757×10^{-1} [D] None

3. Subtract 5481 from 7482

[A] 0.2001 [B] 0.2001×10^3
[C] 0.2001×10^{-3} [D] None

1.2.4 Multiplication operation

In multiplication operation, the mantissas are multiplied and the exponents are added.

In case of multiplication operation, the mantissa of product will not be in normalized form.

For example, if 0.1234E5 and 0.1111E13 are to be multiplied the we directly multiply the mantissa and add the exponents.

i.e. 0.1234E5 * 0.1111E13 = (0.1234 * 0.1111) E (5+13)

Example 1.6

Multiply 0.3754E5 by 0.8576E4

Solution:

0.3754E5 × 0.8576E4 = (0.3754 × 0.8576) E (5+4)

= 0.32194304E9

Thus, we obtain 0.1475E8 after truncating the mantissa of the product to four digits. And this product is already in the normalized form.

Example 1.7

Multiply 0.2345E5 by 0.1111E15

Solution:

0.2345E5 ×0.1111E15 = (0.2345 × 0.1111)E(5+15)

=0.02605295E20

Thus, we obtain 0.0260E20 after truncating the mantissa of the product to four digits. And since the product is less than 0.1, the decimal point is shifted one position to the right and the exponent is decreased by 1. The resultant product becomes 0.2605E19.

1.2.5 Division operation

In Division Operation, the mantissa of first number is divided by the mantissa of the second number and the exponent of second number is subtracted from the exponent of first number. Note, that the mantissa of the quotient will not be normalized.

Further Note that the magnitude of the quotient may become greater than 1.0, but will always be less than 10.0. Therefore, at most the decimal point of mantissa of the quotient will be shifted one position to the left and the exponent will be increased by 1. And as a result of the increase, the exponent can increase up to +99. Now if the mantissa is negative, this results in underflow or overflow.

Example 1.8

Divide 0.9876E-5 by 0.1231E99

Solution:

0.9876E-5 ÷ 0.1231E99 = (0.9876 ÷ 0.1231)E(-5-99)

= 8.02274574E-104

The mantissa of the quotient is greater than 1.0 therefore the decimal point is shifted one position to the left and the exponent is increased by 1. The resultant quotient becomes 0.8022E-103.

Example 1.9

Divide 0.9876E5 by 0.1231E-99

Solution:

0.9876E5 ÷ 0.1231E-99 = (0.9876 ÷ 0.1231)E(5-(-99))

= 8.02274574E104

The mantissa of the quotient is greater than 1.0 therefore the decimal point is shifted one position to the left and the exponent is increased by 1. The resultant quotient becomes 0.8022E105.

Check Your Progress-2

1. Compute $\frac{0.0785}{3580} = \frac{0.785\times10^{-1}}{0.385\times10^{4}}$

[A] 0.219×10^{-4} [B] 0.219×10^{4}

[C] 0.253×10^{5} [D] 0.253×10^{-5}

2. Divide 0.10000 E 99 to 0.6457 E -6

[A] 0.6457 E -105 [B] 0.6457 E 105

[C] 0.1548 E -105 [D] 0.1548 E 105

3. If we multiply 0.1237 E 51 to 0.5286 E 55 then result obtained would

[A] Overflow [B] Underflow

[C] Can't be said [D]None of the Above

1.3 LIMITATION OF FLOATING-POINT REPRESENTATION:

Given below are major limitations of floating-point representation:

- Floating point representation is basically complex representation as it uses two fields: mantissa and exponent.
- Length of registers for storing floating point numbers is large.

1.4 LET US SUM UP

In this unit, we:

- Discussed number systems and floating-point representation
- Explained floating-point arithmetic

1.5 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. [A] 0.2909×10^3
2. [A] 0.139×10^1
3. [B] 0.2001×10^3

Check Your Progress-2

1. [A] 0.219×10^{-4}
2. [A] 0.6457 E -105
3. [A] Overflow

1.6 GLOSSARY

1. **Operation** is a function which takes zero or more input values to obtain well-defined output value.
2. **Arithmetic Operation** is part of mathematics that involves the study of numbers and the operation of number that are useful in all other fields of mathematics.
3. **Mantissa** is the part of logarithm after the decimal point.
4. **Error** is the difference between the true value and the approximation of that value.

1.7 ASSIGNMENT

1. What is normalized floating-point representation? Express the number 0.000008754 in the normalized floating-point representation.
2. (i) Divide 0.9854 E 5 by 0.1987 E 3
   (ii) Multiply 0.6585 E 56 by 0.4375 E -53
   (iii) Subtract 0.3846 E 4 by 0.1736 E 4
   (iv) Add 0.7364 E 2 by 0.7686 E 4

1.8 ACTIVITY

1. Find the value of (6 × 4) ÷ 12 + 72 ÷ 8 – 9

2. Find the value of 108 ÷ 3 + (5 × 2) - 36

1.9 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 2: ERRORS

Unit Structure

2.0 LEARNING OBJECTIVES:

Whenever a numerical method is applied to the problem, then the numerical result obtained is certainly approximate, i.e. it contains some errors.

From this chapter students will learn and understand errors occurring in numerical computation.

2.1 ERRORS

Whenever a numerical operation is applied to a problem, then the result obtained is certainly approximate value, i.e. the value obtained contains some errors.

An error is defined as the difference between the actual value of the problem and the value obtained from the numerical operation. Consider x represents some quantity and x_a is approximations to x. Then;

Error = actual value – approximate value

= x - x_a

2.1.1 Errors in Computation

Computational errors are mostly occurred due to following reasons:

- Normalized floating-point representation
- Truncation of infinite series expansion
- Inefficient algorithm

As we have earlier seen in previous chapter that while performing different arithmetic operations using normalized floating-point representation, some digits have to be shorten in order to fit it in normalized float-point form. And due to such shortenings various types of errors occur depending upon size of the numbers and the type of operations used.

Similarly, errors are introduced because of infinite series or process, such as sin(x), cos(x), log(x), e^x, etc.
For example, the following infinite series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Is used to find the value of sin(x). Since the series is infinite, it is not possible to use all the terms in computations. In such cases the process is executed till a finite number of terms and then terminated.
And because of the terms omitted due to termination will create error in the computed result.

2.1.2 Truncation And Round-off errors

Truncation Error:

In truncation errors, some digits are excluded from the number. There are mainly two situations when

i) Numbers are represented in normalized floating-point form. The mantissa can accommodate only a few digits, for example, only four in our hypothetical computer. In our hypothetical computer, 0.0356879 would take the form 0.3568E-1 in normalized floating-point representation. The digits 7 and 9 have been discarded. This truncation introduces errors in the input data.

ii) A number is converted from one system to another. For example, in binary, 13.1 corresponds to 1101.0001100110011... There is a repeating fraction in this number, which means the conversion must be terminated after some digits. This introduces the truncation error.

Round-off Error:

Round-off errors occur when floating-point numbers are chopped, rounded, or used in arithmetic operations.

Computers can only represent real numbers with a fixed precision of mantissa. A computer's representation of true values may not always be accurate. This is called round-off error. It is possible that round-off errors are the most troublesome, since they can accumulate and cause significant problems.

Check Your Progress-1

1. Let x = 0.00573735. Find the absolute error if x is truncated to the three decimal points.
2. The solution of a problem is given as 5.284. It is known that the absolute error in the solution is less than 0.01 .Find the interval within which the exact value must lie.

2.1.3 Absolute and Relative errors

Absolute Error:

Absolute Error is the modulus of the difference between the actual value and the approximate value of the given problem. If x is the actual value and x_a is the approximate value of the problem then the Absolute Error is given by:

$$\text{Absolute Error} = | x - x_a |$$

Relative Error:

Relative Error is the ratio of the error to the actual value of the problem. If x is the actual value and x_a is the approximate value of the problem, then the relative error is given by:

$$\text{Relative Error} = \frac{x - x_a}{x}$$

Example 2.1

Let x = 0.00473817. Find the absolute error if x is truncated to three decimal point.

Solution:

Given, $x = 0.00473819$

$= 0.473819\times10^{-2}$

$x_a = 0.473\times10^{-2}$

error $= x - x_a$

$= 0.473819\times10^{-2} - 0.473\times10^{-2}$

Hence

$| x - x_a | = 0.000473\times10^{-2}$

$= 0.473\times10^{-5}$

Which is less than 10^{-2-3}.

Example 2.2

Given the solution of a problem as $x_0 = 46.93$ with the relative error in the solution at most 4%. Find, to four decimal digits, the range of values within which the exact value the solution must lie.

Solution:

Given maximum relative error = 0.04
Therefore

$$-0.04 < \frac{x - x_0}{x} < 0.04$$

If $\frac{x - x_0}{x} < 0.04$

Then, $x - x_0 < 0.04x$

$\Rightarrow x(1-0.04) < x_a$

$\Rightarrow x < \frac{x_0}{(1-0.04)}$

$= \frac{46.93}{0.96} = 48.8854167$

However, if $-0.04 < \frac{x - x_0}{x}$

Then $x - x_0 > -0.04x$

$\Rightarrow x(1+0.04) > x_0$

$\Rightarrow x > \frac{x_0}{1+0.04}$

$= \frac{46.93}{1.04} = 45.125$

Hence,

$$45.125 < x < 48.8854167$$

Check Your Progress-2

1. Let the exact value of a number be 7.745. If it is round-off to two decimal digits, find the value of absolute and relative error and also write relative percentage.
2. Find the relative percentage error in approximate representation of 4/3 by 1.33 .

2.2 LET US SUM UP

In this unit, we:

- Learn how errors are computed and how it can be prevent
- Learned different types of errors and how to calculate them.

2.3 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. 0.573×10^{-5}

2. (5.274, 5.294)

Check Your Progress-2

1. 0.5%
2. 0.0645%

2.4 GLOSSARY

1. **Truncation** means to be shortened off.

2. **Inefficient** means not working or producing in best way possible

3. **Error** is the difference between the true value and the approximation of that value.

4. **Modulus** is the of the positive value of any problem.

2.5 ASSIGNMENT

1. Write down approximate value of $\frac{\pi}{4}$ correct to four significant digits and then find (a) absolute error (b) relative error (c) relative percentage.

2. If $\delta x = 0.006$ and $\delta y = 0.003$ be the absolute errors in x = 2.56 and y = 5.24, find the relative error in computation of x + y.

3. Let x = 0.0083465, find the relative error if
   (i) x is rounded off to three decimal digits
   (ii) If x is chopped off to three decimal digits

2.6 ACTIVITY

1. A thermometer is calibrated 150^{0}C to 200^{0}C. The accuracy is specified within ±0.25%. What is the maximum static error?
2. A Calculator has a capacity to calculate numbers up to 12 digits. And the accuracy is specified within ±0.20%. What is the maximum error possible?

2.7 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age international publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 3: SOLVING NON-LINEAR EQUATIONS

Unit Structure

3.0 LEARNING OBJECTIVES

In this unit we will learn variety of methods to find approximations for the roots of an equations. And we will also learn methods use to find solutions of algebraic and Transcendental Equations which are represented by f(x) =0.

Iteration is the fundamental principle in computer science. As the name says iteration means that the process is repeated until an answer is achieved. This technique is used to roots of equation, solution of linear equation and nonlinear system of equations.

3.1 INTRODUCTION

Finding solution of non-linear equations is of interest not only to mathematicians but also to scientist and engineers. Though most of the non-linear equation can be solved algebraically i.e. analytically. But there are many non-linear equations which cannot be solved algebraically.

For example;

$$3^{2x} + 3x - e^{x} = 0$$

It seems very simple but cannot be solved algebraically.

The solution obtained by algebraic manipulation are known as algebraic solutions or analytical solutions. There are many examples in which algebraic solutions of non-linear equations does not exist but is extremely complicated and impractical for day-to-day purposes. In this chapter we will learn some methods to find solutions of non-linear equations, which will be numerical and not algebraic, and called numerical solutions.

3.2 TYPES OF NON-LINEAR EQUATIONS

3.2.1 Polynomial equations

The polynomials are frequently occurring functions in science and engineering field.

A polynomial equation can be written in following general form:

$$a_n x^{n +} a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x + a_0 = 0 \quad \text{where } a_n \neq 0$$

It is a n^{th} degree polynomial in x and has n roots. These roots may be

- Real and Different
- Real and Equal
- Complex

Note: Complex numbers appear in pairs and are in form of a+ib and a-ib where i=……., ia an imaginary number, and a and b are real numbers and represent real and imaginary parts of root.

3.2.2 Transcendental equations

Non-polynomial equations are called transcendental equations. Some examples of transcendental equations are

$$xe^x - x\sin x = 0$$

$$e^x\cos x - 3x = 0$$

$$e^{3x} - \frac{x}{2} = 0$$

$$2^x - x - 3 = 0$$

Note:

A Transcendental equation may have finite/infinite number of real roots or may not have any real root at all.

Check Your Progress-1

1. Find the approximate value of a real root of

$$f(x) = x\log_{10}x - 1.2 = 0$$

2. Find the approximate value of a real root of the equation

$$\sin x - x^3 - 1 = 0$$

3.3 METHODS OF SOLVING NON-LINEAR EQUATIONS

In general, there are two kind of methods to find solution of non-linear equations. These are:

1. Direct methods
2. Iterative methods

3.3.1 Direct methods

Direct method gives the roots of non-linear equations in a few steps and this methos is also capable to give all the roots at the same time.

For example, the root of quadratic equation:

$$ax^2 + bx + c = 0$$

Where a ≠ 0 are given by

$$X_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.3.2 Iterative methods

Iterative methods are also known as trial-and-error methods. An iterative method involves repeating approximations in order to arrive at a result. A series of approximations are obtained by repeating a fixed sequence of steps until the solution is accurate enough.

Generally, through Iterative method we only get one root at a time. And this method is very difficult and time consuming for solving the non-linear equation manually. However, it is suitable for use on computer because of following reasons:

1. Iterative methods can be concisely expressed as computational algorithms.
2. Round-off error are negligible in iterative methods as compared to direct methods.
3. It is possible to formulate algorithms which can handle class of similar problems.

3.4 ITERATIVE METHODS

With this knowledge, Let's discuss some very well-known iterative methods to find solution of the algebraic and transcendental equations.

3.4.1 Bisection method

This is one of the simplest iterative methods. Bisection method is also known as Bolzano method or Interval method. To start with two initial approximations x_0 and x_1 such that $f(x_0)f(x_1) < 0$ which means $f(x_0)$ and $f(x_1)$ have opposite signs, which ensures that the next root lies between x_0 and x_1, then next root say x_2, is the midpoint of the interval $[x_0, x_1]$ is computed.
Now,

If $f(x_2) = 0$, then we get the root at x_2.

If $f(x_0)$ and $f(x_2)$ are of opposite sign, then the root lies between the interval (x_0, x_2). Thus, x_1 is replaced by x_2, and the new interval is formed, which is the half of the previous interval. And this interval is again bisected.

If $f(x_1)$ and $f(x_2)$ are of opposite sign, then the root lies between the interval (x_2, x_1). Thus, x_0 is replaced by x_2, and the new interval is formed, which is the half of the previous interval. And this interval is again bisected.

Similarly, this procedure is continued until a root of desired accuracy is obtained.

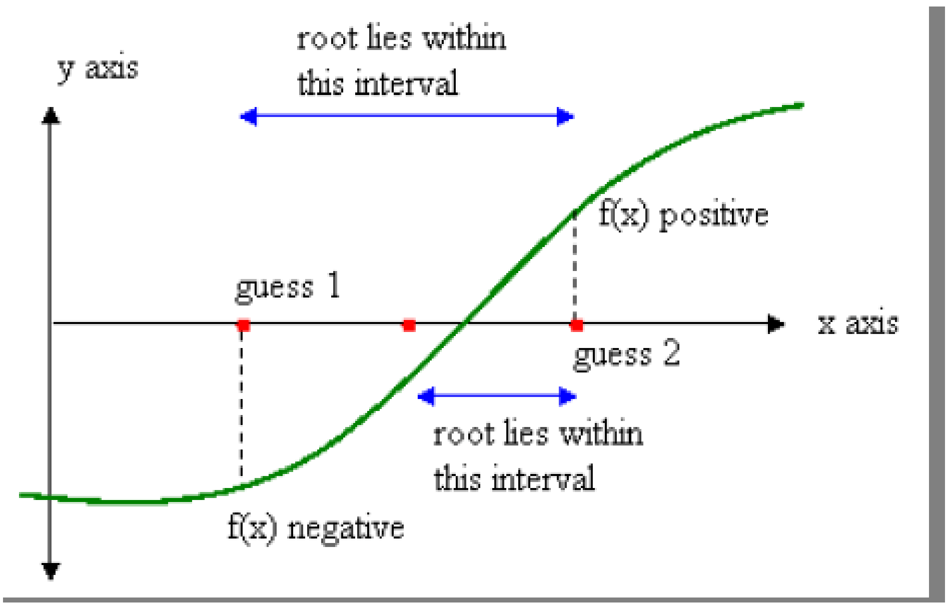

Figure3.1 Root approximation by approximation method

Example 3.1

Find a root of an equation $f(x)=2x^3-2x-5$ using Bisection method

Solution:
Here $2x^3-2x-5=0$
Let $f(x)=2x^3-2x-5$

Here

x	0	1	2
$f(x)$	-5	-5	7

1*st* iteration:

Here $f(1)=-5<0$ and $f(2)=7>0$

∴ Now, Root lies between 1 and 2

$x_0 = (1+2)/2 = 1.5$

$f(x_0) = f(1.5) = 2 \cdot 1.53 - 2 \cdot 1.5 - 5 = -1.25<0$

2*nd* iteration :

Here $f(1.5)=-1.25<0$ and $f(2)=7>0$

∴ Now, Root lies between 1.5 and 2

$x_1 = (1.5+2)/2 = 1.75$

$f(x_1) = f(1.75) = 2 \cdot 1.753 - 2 \cdot 1.75 - 5 = 2.2188>0$

3*rd* iteration :

Here $f(1.5) = -1.25<0$ and $f(1.75) = 2.2188>0$

∴ Now, Root lies between 1.5 and 1.75

x_2 = (1.5+1.75)/2 = 1.625

$f(x_2)$ = f(1.625) = 2·1.6253-2·1.625-5 = 0.332>0

<u>4*th* iteration :</u>

Here f(1.5) = -1.25<0 and f(1.625) = 0.332>0

∴ Now, Root lies between 1.5 and 1.625

x_3 = (1.5+1.625)/2 = 1.5625

$f(x_3)$ = f(1.5625) = 2·1.56253-2·1.5625-5 = -0.4956<0

<u>5*th* iteration :</u>

Here f(1.5625) = -0.4956<0 and f(1.625) = 0.332>0

∴ Now, Root lies between 1.5625 and 1.625

x_4 = (1.5625+1.625)/2 = 1.5938

$f(x_4)$ = f(1.5938) = 2·1.59383-2·1.5938-5 = -0.0911<0

<u>6*th* iteration :</u>

Here f(1.5938) = -0.0911<0 and f(1.625) = 0.332>0

∴ Now, Root lies between 1.5938 and 1.625

x_5 = (1.5938+1.625)/2 = 1.6094

$f(x_5)$ = f(1.6094) = 2·1.60943-2·1.6094-5 = 0.1181>0

<u>7*th* iteration :</u>

Here f(1.5938) = -0.0911<0 and f(1.6094) = 0.1181>0

∴ Now, Root lies between 1.5938 and 1.6094

x_6 = (1.5938+1.6094)/2 = 1.6016

$f(x_6)$ = f(1.6016) = 2·1.60163-2·1.6016-5 = 0.0129>0

<u>8*th* iteration :</u>

Here f(1.5938) = -0.0911<0 and f(1.6016) = 0.0129>0

∴ Now, Root lies between 1.5938 and 1.6016

x_7 = (1.5938+1.6016)/2 = 1.5977

$f(x_7)$ = f(1.5977) = 2·1.59773-2·1.5977-5 = -0.0393<0

<u>9*th* iteration :</u>

Here f(1.5977) = -0.0393<0 and f(1.6016) = 0.0129>0

∴ Now, Root lies between 1.5977 and 1.6016

x_8 = (1.5977+1.6016)/2 = 1.5996

$f(x_8)$ = f(1.5996) = 2·1.59963-2·1.5996-5 = -0.0132<0

<u>10*th* iteration :</u>

Here f(1.5996) = -0.0132<0 and f(1.6016) = 0.0129>0

∴ Now, Root lies between 1.5996 and 1.6016

x_9 = (1.5996+1.6016)/2 = 1.6006

$f(x_9)$ = f(1.6006) = 2·1.60063-2·1.6006-5 = -0.0002<0

Approximate root of the equation $2x^3-2x-5=0$ using Bisection method is 1.6006

Example 3.2

Find a root of an equation $f(x)$=2x3-2x-5 using Bisection method

Solution:
Here 2x3-2x-5=0
Let $f(x)$=2x3-2x-5

Here

x	0	1	2
***f*(*x*)**	-5	-5	7

<u>1*st* iteration :</u>

Here f(1)=-5<0 and f(2)=7>0

∴ Now, Root lies between 1 and 2

x_0 = (1+2)/2 = 1.5

$f(x_0)$ = f(1.5) = 2·1.53-2·1.5-5 = -1.25<0

<u>2*nd* iteration :</u>

Here f(1.5) = -1.25<0 and f(2) = 7>0

∴ Now, Root lies between 1.5 and 2

x_1 = (1.5+2)/2 = 1.75

$f(x_1)$ = f(1.75) = 2·1.753-2·1.75-5 = 2.2188>0

3*rd* iteration :

Here $f(1.5) = -1.25<0$ and $f(1.75) = 2.2188>0$

∴ Now, Root lies between 1.5 and 1.75

$x_2 = (1.5+1.75)/2 = 1.625$

$f(x_2) = f(1.625) = 2\cdot1.6253-2\cdot1.625-5 = 0.332>0$

4*th* iteration :

Here $f(1.5) = -1.25<0$ and $f(1.625) = 0.332>0$

∴ Now, Root lies between 1.5 and 1.625

$x_3 = (1.5+1.625)/2 = 1.5625$

$f(x_3) = f(1.5625) = 2\cdot1.56253-2\cdot1.5625-5 = -0.4956<0$

5*th* iteration :

Here $f(1.5625) = -0.4956<0$ and $f(1.625) = 0.332>0$

∴ Now, Root lies between 1.5625 and 1.625

$x_4 = (1.5625+1.625)/2 = 1.5938$

$f(x_4) = f(1.5938) = 2\cdot1.59383-2\cdot1.5938-5 = -0.0911<0$

6*th* iteration :

Here $f(1.5938) = -0.0911<0$ and $f(1.625) = 0.332>0$

∴ Now, Root lies between 1.5938 and 1.625

$x_5 = (1.5938+1.625)/2 = 1.6094$

$f(x_5) = f(1.6094) = 2\cdot1.60943-2\cdot1.6094-5 = 0.1181>0$

7*th* iteration :

Here $f(1.5938) = -0.0911<0$ and $f(1.6094) = 0.1181>0$

∴ Now, Root lies between 1.5938 and 1.6094

$x_6 = (1.5938+1.6094)/2 = 1.6016$

$f(x_6) = f(1.6016) = 2\cdot1.60163-2\cdot1.6016-5 = 0.0129>0$

8*th* iteration :

Here $f(1.5938) = -0.0911<0$ and $f(1.6016) = 0.0129>0$

∴ Now, Root lies between 1.5938 and 1.6016

$x_7 = (1.5938+1.6016)/2 = 1.5977$

$f(x_7)$ = f(1.5977) = 2·1.59773-2·1.5977-5 = -0.0393<0

<u>9*th* iteration :</u>

Here f(1.5977) = -0.0393<0 and f(1.6016) = 0.0129>0

∴ Now, Root lies between 1.5977 and 1.6016

x_8 = (1.5977+1.6016)/2 = 1.5996

$f(x_8)$ = f(1.5996) = 2·1.59963-2·1.5996-5 = -0.0132<0

<u>10*th* iteration :</u>

Here f(1.5996)=-0.0132<0 and f(1.6016)=0.0129>0

∴ Now, Root lies between 1.5996 and 1.6016

x_9 = (1.5996+1.6016)/2 = 1.6006

$f(x_9)$ = f(1.6006) = 2·1.60063-2·1.6006-5 = -0.0002<0

Approximate root of the equation $2x^3-2x-5=0$ using Bisection method is 1.6006

3.4.2 False position method

The false position method is also known as Regula-Falsi or the method linear interpolation. It is similar to bisection method but it's faster than it. This method also starts with two initial approximations a and b for which f(a)f(b) < 0, i.e. f(a) is positive and f(b) is negative.

After assuming the two points. Join the points (a, f(a)) and (b, f(b)) by a straight line. And the point where the line intersects the x-axis (Say c) is the next point of approximations.

Then there are three possibilities:

I. If f(c) = 0, then c is a root of the equation f(x) = 0
II. If f(a) and f(c) are of opposite sign, then the root lies between the interval (a, c). So, b is replaced by c and the iterative procedure is repeated.
III. If f(a) and f(c) are of same sign, then the root lies between the interval (c, b). So, a is replaced by c and the iterative procedure is repeated.

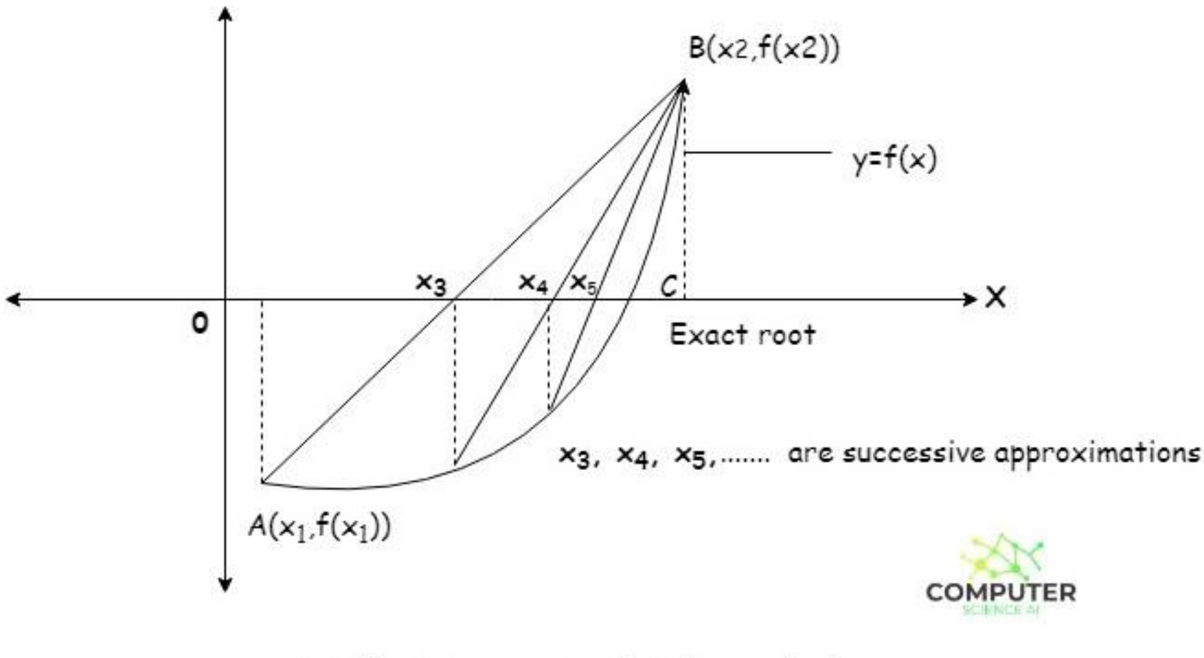

Fig Root approximation by false position method

Example 3.3

Find a root of an equation $f(x)= x^3 -x-1$ using False Position method

Solution:

Here x^3-x-1=0

Let $f(x)= x^3- x- 1$

Here

x	0	1	2
$f(x)$	-1	-1	5

1*st* iteration :

Here $f(1) = -1<0$ and $f(2) = 5>0$

∴ Now, Root lies between $x_0 = 1$ and $x_1 = 2$

$x_2 = x_0 - f(x_0)\cdot x_1 – x_1f(x_1) - f(x_0)$

x_2 = 1-(-1)·2-15-(-1)

x_2 = 1.1667

$f(x_2) = f(1.1667)$ = 1.16673-1.1667-1 = -0.5787<0

2*nd* iteration :

Here $f(1.1667) = -0.5787<0$ and $f(2) = 5>0$

∴ Now, Root lies between $x_0 = 1.1667$ and $x_1 = 2$

$x_3 = x_0 - f(x_0)\cdot x_1 - x_0f(x_1) - f(x_1)$

x_3 = 1.1667-(-0.5787)·2-1.16675-(-0.5787)

x_3 = 1.2531

$f(x_3) = f(1.2531)$ = 1.25313-1.2531-1 = -0.2854<0

3*rd* iteration :

Here $f(1.2531) = -0.2854<0$ and $f(2) = 5>0$

∴ Now, Root lies between $x_0 = 1.2531$ and $x_1=2$
$x_4 = x_0 - f(x_0)\cdot x_1 – x_0 f(x_1)-f(x_0)$
$x_4 = 1.2531-(-0.2854)\cdot 2-1.25315-(-0.2854)$
$x_4 = 1.2934$
$f(x_4) = f(1.2934) = 1.29343-1.2934-1 = -0.1295<0$

<u>4*th* iteration :</u>
Here $f(1.2934)=-0.1295<0$ and $f(2)=5>0$
∴ Now, Root lies between $x_0=1.2934$ and $x_1 = 2$
$x_5 = x_0 - f(x_0)\cdot x_1 - x_0 f(x_1)-f(x_0)$
$x_5 = 1.2934-(-0.1295)\cdot 2-1.29345-(-0.1295)$
$x_5 = 1.3113$
$f(x_5) = f(1.3113) = 1.31133-1.3113-1 = -0.0566<0$

<u>5*th* iteration :</u>
Here $f(1.3113) = -0.0566<0$ and $f(2) = 5>0$
∴ Now, Root lies between $x_0 = 1.3113$ and $x_1 = 2$
$x_6 = x_0 - f(x_0)\cdot x_1 – x_0 f(x_1) - f(x_0)$
$x_6 = 1.3113-(-0.0566)\cdot 2-1.31135-(-0.0566)$
$x_6 = 1.319$
$f(x_6) = f(1.319) = 1.3193-1.319-1 = -0.0243<0$

<u>6*th* iteration :</u>
Here $f(1.319)=-0.0243<0$ and $f(2)=5>0$
∴ Now, Root lies between $x_0 = 1.319$ and $x_1 =2$
$x_7 = x_0 -f(x_0)\cdot x_1 - x_0 f(x_1)-f(x)$
$x_7 = 1.319-(-0.0243)\cdot 2-1.3195-(-0.0243)$
$x_7 = 1.3223$
$f(x_7) = f(1.3223) = 1.32233-1.3223-1 = -0.0104<0$

<u>7*th* iteration :</u>
Here $f(1.3223) = -0.0104<0$ and $f(2) = 5>0$
∴ Now, Root lies between $x_0 = 1.3223$ and $x_1 = 2$
$x_8 = x_0 - f(x_0)\cdot x_1 - x_0 f(x_1)-f(x_1)$
$x_8 = 1.3223-(-0.0104)\cdot 2-1.32235-(-0.0104)$
$x_8 = 1.3237$
$f(x_8) = f(1.3237) = 1.32373-1.3237-1 = -0.0044<0$

<u>8*th* iteration :</u>
Here $f(1.3237)=-0.0044<0$ and $f(2)=5>0$
∴ Now, Root lies between $x_0=1.3237$ and $x_1=2$
$x_9 = x_0-f(x_0)\cdot x_1 – x_0 f(x_1) - f(x_0)$
$x_9 = 1.3237-(-0.0044)\cdot 2-1.32375-(-0.0044)$
$x_9 = 1.3243$
$f(x_9) = f(1.3243) = 1.32433-1.3243-1 = -0.0019<0$

<u>9*th* iteration :</u>
Here $f(1.3243) = -0.0019<0$ and $f(2) = 5>0$
∴ Now, Root lies between $x_0=1.3243$ and $x_1 =2$
$x_{10} =x_0 -f(x_0)\cdot x_1 - x_0 f(x_1)-f(x_0)$

x_{10} = 1.3243-(-0.0019)·2-1.32435-(-0.0019)
x_{10} = 1.3245
$f(x_{10})$ = f(1.3245) = 1.32453-1.3245-1 = -0.0008<0

10*th* iteration :
Here f(1.3245)=-0.0008<0 and f(2)=5>0
∴ Now, Root lies between x_0 = 1.3245 and x_1 = 2
$x_{11} = x_0 - f(x_0)\cdot x_1 - x_0 f(x_1) - f(x_0)$
x_{11} = 1.3245-(-0.0008)·2-1.32455-(-0.0008)
x_{11} = 1.3246
$f(x_{11})$ = f(1.3246) = 1.32463-1.3246-1 = -0.0003<0

Approximate root of the equation x^3-x-1=0 using False Position method is 1.3246

3.4.3 Secant method

A secant method is similar to the false position method with only one difference: instead of using only approximations that bound the interval containing the root, two recent approximations are used instead. There is also no need to enclose the root in the initial approximations. The secant method is faster than the false position method, however it does not always guarantee convergence.

The system's work is described by assuming x_0, and x_1 are two initial approximations. Using a straight line, connect the points (x_0, $f(x_0)$) and (x_1, $f(x_1)$). Taking the intersection of this line with the X-axis, we can estimate the root of the equation at that point. Assume that this point of intersection is x_2. For the next iteration, we will use x_1 and x_2 as approximations. The line is drawn joining the points (x_1, $f(x_1)$) and (x_2, $f(x_2)$). We will use x_3 as the point of intersection.

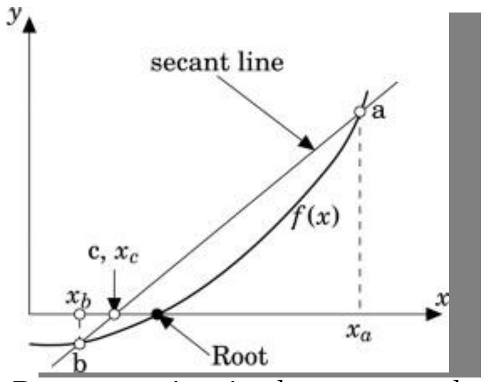

Fig. Root approximation by secant method

Example 3.5

Find a root of an equation $f(x)=7x^3 - 5x$ using Secant method

Solution:
Here $7x^3 - 5x=0$
Let $f(x) = 7x^3-5x$

Here

x	0	1
$f(x)$	0	2

Here $f(0)=0$

∴ Root of the equation $7x^3 - 5x$ is 0.

Example 3.6

Find a root of an equation $f(x) = 4x^3-5x + 9$ using Secant method

Solution:

Here $4x^3-5x+9=0$

Let $f(x)=4x^3-5x+9$

Here

x	0	-1	-2	-3
$f(x)$	9	10	-13	-84

1*st* iteration :

$x_0 = -2$ and $x_1 = -1$

$f(x_0) = f(-2) =-13$ and $f(x_1) = f(-1) =10$

$\therefore x_2 = x_0 - f(x_0)\cdot x1 - x_0 f(x_1) - f(x_0)$

$x_2 = -2-(-13)\cdot -1-(-2)10-(-13)$

$x_2 = -1.4348$

$\therefore f(x_2) = f(-1.4348) = 4\cdot(-1.4348)3-5\cdot(-1.4348) + 9 = 4.3593$

2*nd* iteration :

$x_1 = -1$ and $x_2 = -1.4348$

$f(x_1) = f(-1) = 10$ and $f(x_2) = f(-1.4348) = 4.3593$

$\therefore x_3 = x_1 - f(x_1)\cdot x_2 - x_1 f(x_2) - f(x_1)$

$x_3 = -1-10\cdot -1.4348- (-1)4.3593 -10$

$x_3 = -1.7708$

$\therefore f(x_3) = f(-1.7708) = 4\cdot(-1.7708)3-5\cdot(-1.7708) + 9 = -4.357$

3*rd* iteration :

$x_2 = -1.4348$ and $x_3 = -1.7708$

$f(x_2) = f(-1.4348) = 4.3593$ and $f(x_3) = f(-1.7708) = -4.357$

$\therefore x_4 = x_2 - f(x_2)\cdot x_3 - x_2 f(x_3) - f(x_2)$

$x_4 = -1.4348 - 4.3593\cdot -1.7708 - (-1.4348) - 4.357 - 4.3593$

$x_4 = -1.6028$

$\therefore f(x_4) = f(-1.6028) = 4\cdot(-1.6028)3 -5\cdot(-1.6028) + 9 = 0.5429$

4*th* iteration :
x_3 = -1.7708 and x_4 = -1.6028
$f(x_3)$ = f(-1.7708) = -4.357 and $f(x_4)$ = f(-1.6028) = 0.5429

∴$x_5 = x_3 - f(x_3)\cdot x_4 – x_3 f(x_4) - f(x_3)$
x_5 = -1.7708 -(-4.357)·-1.6028 -(-1.7708)0.5429 -(-4.357)
x_5 = -1.6214
∴$f(x_5)$ = f(-1.6214) = 4·(-1.6214)3-5·(-1.6214) + 9 = 0.0555

5*th* iteration :
x_4 = -1.6028 and x_5 = -1.6214
$f(x_4)$ = f(-1.6028) = 0.5429 and $f(x_5)$ = f(-1.6214) = 0.0555

∴$x_6 = x_4 - f(x_4)\cdot x_5 – x_4 f(x_5) - f(x_4)$
x_6 = -1.6028 - 0.5429·-1.6214 - (-1.6028)0.0555 -0.5429
x_6=-1.6236
∴$f(x)$ = f(-1.6236) = 4·(-1.6236)3-5·(-1.6236)+9 = -0.0009

6*th* iteration :

x_5 = -1.6214 and x_6 = -1.6236
$f(x_5)$ = f(-1.6214) = 0.0555 and $f(x_6)$ = f(-1.6236) = -0.0009

∴x_7=$x_5 - f(x_5)\cdot x_6 - x_5 f(x_6) - f(x_5)$
x_7 = -1.6214-0.0555·-1.6236-(-1.6214)-0.0009-0.0555
x_7 = -1.6235
∴$f(x_7)$ = f(-1.6235) = 4·(-1.6235)3-5·(-1.6235)+9 = 0

Approximate root of the equation $4x^3$-5x+9=0 using Secant method is -1.6235

3.4.4 Newton-Raphson method

As one of the fastest iterative methods, Newton-Raphson's method of tangents begins with one approximation and then continues to iterate. In this case, one must choose the initial approximation carefully since the accuracy of the solution will vary greatly based on the initial approximation. Once a proper choice of the initial approximation has been made, it will converge faster than the secant method and false position method.

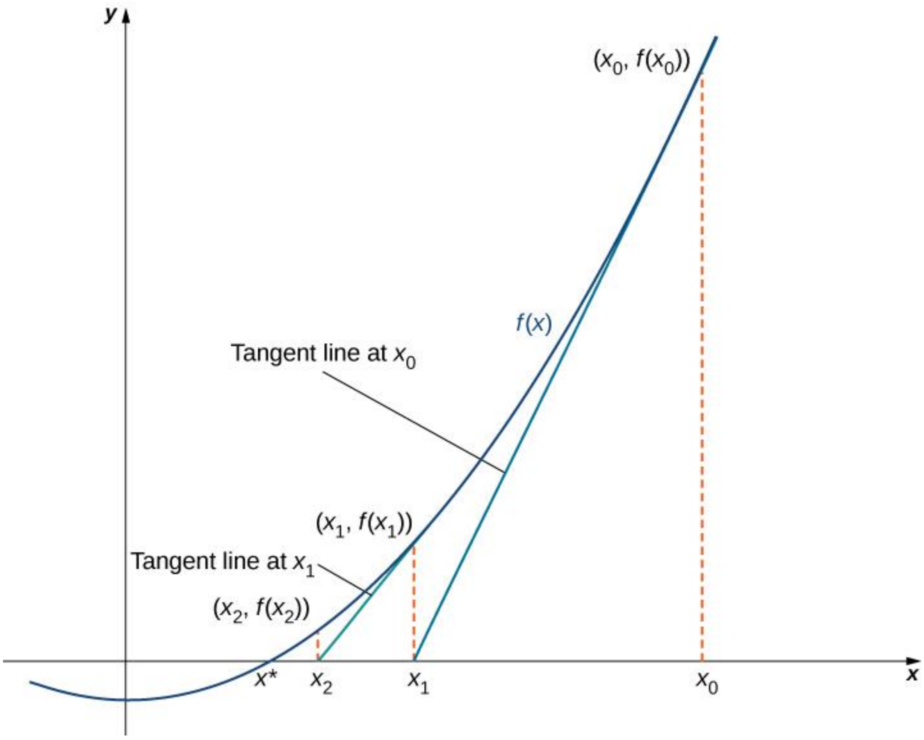

Fig. Root approximation For Newton-Raphson's

Example3.7

Find a root of an equation $f(x)=7x^2 - 4x -7$ using Newton Raphson method

Solution:

Here $7x^2-4x-7=0$

Let $f(x) = 7x^2-4x-7$

$$\frac{d}{dx}(7x^2 - 4x - 7) = 14x - 7$$

$\therefore f'(x) = 14x - 4$

Here

x	0	-1	-2
***f*(*x*)**	-7	4	29

Here $f(-1) = 4 > 0$ and $f(0) = -7 < 0$

$\therefore$ Root lies between -1 and 0

$x_0 = (-1+0)/2 = -0.5$

$x_0 = -0.5$

1*st* iteration :

$f(x_0) = f(-0.5) = 7\cdot(-0.5)^2 -4\cdot(-0.5) -7 = -3.25$

$f'(x_0) = f'(-0.5) = 14\cdot(-0.5) – 4 = -11$

$x_1 = x_0 - f(x_0)/f'(x_0)$

$x_1 = -0.5- (-3.25)/-11$

$x_1 = -0.7955$

<u>2*nd* iteration :</u>

$f(x_1) = f(-0.7955) = 7\cdot(-0.7955)^2 -4\cdot(-0.7955) -7 = 0.6111$

$f'(x_1) = f'(-0.7955) = 14\cdot(-0.7955)-4 = -15.1364$

$x_2 = x_1 - f(x_1)/f'(x_1)$

$x_2 = -0.7955 – (0.6111)/-15.1364$

$x_2 = -0.7551$

<u>3*rd* iteration :</u>

$f(x_2) = f(-0.7551) = 7\cdot(-0.7551)^2 - 4\cdot(-0.7551) -7 = 0.0114$

$f'(x2) = f'(-0.7551) = 14\cdot(-0.7551) – 4 = -14.5712$

$x_3 = x_2 - f(x_2)/f'(x_2)$

$x_3 = -0.7551 -0.0114 -14.5712$

$x_3 = -0.7543$

<u>4*th* iteration :</u>

$f(x_3) = f(-0.7543) = 7\cdot(-0.7543)^2 - 4\cdot(-0.7543) -7 = 0$

$f'(x_3) = f'(-0.7543) = 14\cdot(-0.7543) -4 = -14.5602$

$x_4 = x_3 - f(x_3)/f'(x_3)$

$x_4 = -0.7543 – (0)/-14.5602$

$x_4 = -0.7543$

Approximate root of the equation $7x^2-4x-7=0$ using Newton Raphson method is -0.7543.

Example 3.8

Find a root of an equation $f(x) = 2\cdot\cos(x) - x$ using Newton Raphson method.

Solution :

Here $2\cos(x)-x=0$

Let $f(x)=2\cos(x)-x$

$$\frac{d}{dx}(2(\cos x) - x) = -2\sin(x) - 1$$

$\therefore f'(x) = -2\sin(x)-1$

Here

x	0	1	2
$f(x)$	2	0.0806	-2.8323

Here $f(1) = 0.0806 > 0$ and $f(2) = -2.8323 < 0$

$\therefore$ Root lies between 1 and 2

$x_0 = (1+2)/2 = 1.5$

$x_0 = 1.5$

<u>1*st* iteration :</u>

$f(x_0) = f(1.5) = 2\cos(1.5) - 1.5 = -1.3585$

$f'(x_0) = f'(1.5) = -2\sin(1.5) – 1 = -2.995$

$x_1 = x_0 - f(x_0)f'(x_0)$

$x_1 = 1.5 – (-1.3585)/-2.995$

$x_1 = 1.0464$

<u>2*nd* iteration :</u>

$f(x_1) = f(1.0464) = 2\cos(1.0464) - 1.0464 = -0.045$

$f'(x_1) = f'(1.0464) = -2\sin(1.0464) – 1 = -2.7313$

$x_2 = x_1 - f(x_1)/f'(x_1)$

$x_2 = 1.0464 - (-0.045)/-2.7313$

$x_2 = 1.0299$

<u>3*rd* iteration :</u>

$f(x_2) = f(1.0299) = 2\cos(1.0299) - 1.0299 = -0.0001$

$f'(x_2) = f'(1.0299) = -2\sin(1.0299) – 1 = -2.7145$

$x_3 = x_2 - f(x_2)/f'(x_2)$

$x_3 = 1.0299 – (-0.0001)/-2.7145$

$x_3 = 1.0299$

Approximate root of the equation $2\cos(x) – x = 0$ using Newton Raphson method is 1.0299.

Check Your Progress-2

1. Find the root of the equation $x^3 – x - 11 = 0$, using the bisection method up to third approximation.
2. Find the positive root of $\sin 2x – x = 0$
3. Compute the real root $x \log_{10} x – 1.2 = 0$ correct to fifth decimal place.
4. using the method of false position find the roots of following equations:
   (i) $x^3 - 2x – 5 = 0$
   (ii) $x^3 – 9x + 1 = 0$

Check Your Progress – 3

1. Using Newton-Raphson method find the positive root of $x^3 + 2x^2 + 10x – 20 = 0$
2. Find the value of 1/31 by Newton-Raphson iteration formula correct to four significant figures.
3. Use secant method to find root of
   $f(x) = x^3 – 5x – 7 = 0$ correct to 3 decimal places.
4. Solve the following problem using secant method, using x_0 and x_1 as indicated
   (i) $x^3 – 5x + 3 = 0$, $x_0 = 1.5$, $x_1 = 2.0$
   (ii) $x^6 - x – 1 = 0$, $x_0 = 2.0$, $x_1 = 1.0$

3.6 LET US SUM UP

In this unit, we:

- Learned Different methods to solve non-linear equations
- Different types of iterative methods and how to terminate them

3.7 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. (2,3)
2. (-1,0)

Check Your Progress-2

1. 2.3125
2. 1.90625
3. 2.74065
4. (i)2.094
   (ii) 2.943

Check Your Progress-3

1. 1.368808
2. x = 0.03226
3. x_0=2.5, x_1=3, x_2=2.7183, x_3=2.7442, x_4=2.747, x_5=2.7474
4. (i) 1.834243
   (ii) x_2=1.016129032, x_3=1.190577769, x_4=1.134724138

3.8 GLOSSARY

1. Transcendental equation are any non-algebraic equations.

2. Convergent means to come to a definite limit after more terms are added.

3.9 Assignment

1. Using the method of iteration, find the roots correct to four decimal places.

   1) $x^3 - x - 1 = 0$ $\quad x_0 = 1.5$
   2) $x^4 - 3x + 5 = 0$ $\quad x_0 = 1.0$
   3) $x = 2\sin x$ $\quad x_0 = 1.5$

2. Find the root of the following equations correct to four decimal places by Newton-Raphson :

   1) $x^4 - x - 10 = 0$
   2) $x^3 - 6x + 4 = 0$
   3) $4x^4 - 9x^3 + 1 = 0$, $x_0 = 2.2$

3. Use secant method to find root of

$f(x) = x^3 - 5x - 7 = 0$ correct to 3 decimal places.

3.10 Activity

1. Find the root of the equation $6x^3 - 7x - 19 = 0$, using the bisection method upto third approximation.
2. Using the method of false position find the roots of following equations:
   (i) $5x^3 - 2x - 9 = 0$
   (ii) $x^3 - 3x + 17 = 0$
3. Using Newton-Raphson method find the positive root of $x^3 + 7x^2 + 13x - 23 = 0$
4. Use secant method to find root of

   $f(x) = x^3 - 3x - 11 = 0$ correct to 3 decimal places.

3.11 Further Reading

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012.

Computer Oriented Numerical Method

BLOCK 2: Solving Simultaneous Linear Algebraic Equations And Interpolation

Block Structure

BLOCK2:

UNIT1 MATIRCES

Objectives, Types of Matrices, Operations for Matrices, Transpose of Matrix, Trace of Matrix, Let us Sum Up

UNIT2 DETERMINANTS

Objectives, Determinant of Matrix, Minors and Co-factors of Determinant, Let us Sum Up

UNIT3 SOLVING SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

Objectives, Solution of Non-homogeneous System of Linear Equation, Direct Method, Iterative Method, Let us Sum Up

UNIT4 INTERPOLATION

Objectives, Lagrangian Interpolation, Finite Differences and Difference Table, Newton's Method of interpolation, Let us Sum Up

UNIT 1 : MATRICES

UNIT STRUCTURE

1.0 INTRODUCTION:

A matrix is arrangement of x*y elements in rectangular form of x rows and y columns which are bounded by brackets [ ]. Such formation is known as x by y matrix, and also written as $x \times y$ matrix.
For example:
$\rightarrow A = [1 \quad 2 \quad 3]$; Here A is a 1×3 matrix.

$\rightarrow B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$; Here B is a 3×1 matrix.

Hence, in general a matrix can be written in form of

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1y} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2y} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3y} \\ a_{41} & a_{42} & a_{43} & . & . & . & a_{4y} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{x1} & a_{x2} & a_{x3} & . & . & . & a_{xy} \end{bmatrix}$$

Here; a_{xy} denotes the element in x^{th} row and the y^{th} column of the matrix A.
Thus the matrix can be also denoted by $[a_{xy}]$.

1.1 TYPES OF MATRICES:

There are many types of matrices, such as

1.1.1 Row Matrix:

A matrix consisting only one row is known as row matrix. And sometimes is also known as row vector.

For Eg $A = [1 \quad 2 \quad 3]$

1.1.2 Column Matrix:

A matrix consisting of only one column is known as column matrix. And sometimes is also known as column vector.

For Eg $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

1.1.3 Square Matrix :

A matrix consisting of same number of columns and rows. Square matrix consisting of n rows and n columns, then it is known as square matrix of order n.

For Eg: $D = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{bmatrix}$

1.1.4 Diagonal Matrix :

A square matrix whose all non-diagonal entries are zero are known as diagonal matrix.

For Eg: $E = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 19 \end{bmatrix}$

1.1.5 Identity Matrix:

A square matrix whose all non-diagonal entries are zero and all diagonal are one are known as Identity Matrix.

For Eg: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1.1.6 Symmetric Matrix:

In a square matrix if;

$a_{xy} = a_{yx}$ for all values of x and y

then the matrix is known as Symmetric Matrix.

For Eg: $A = \begin{bmatrix} 1 & 4 & 7 & 3 \\ 4 & 9 & 5 & 8 \\ 7 & 5 & 11 & 2 \\ 3 & 8 & 2 & 1 \end{bmatrix}$

1.1.7 Skew-symmetric Matrix:

In a square matrix if;

$a_{xy} = -a_{yx}$ for all values of x and y

and all the diagonal values of matrix is zero, then the matrix is known as Skew-symmetric Matrix.

For Eg: $A = \begin{bmatrix} 0 & 4 & 7 & 3 \\ -4 & 0 & -5 & 8 \\ 7 & 5 & 0 & -2 \\ -3 & -8 & 2 & 0 \end{bmatrix}$

1.2 OPERATIONS FOR MATRICES:

1.2.1 Addition Operation:

If there are two matrices, say A and B, are of same order then the sum of the two matrices can be written in form of "A+B " , and their corresponding elements can be written in form of

$d_{xy} = a_{xy} + c_{xy}$ for all the values of x and y

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Then,

$$A + C = \begin{bmatrix} a_{11}+c_{11} & a_{12}+c_{12} & a_{13}+c_{13} \\ a_{21}+c_{21} & a_{22}+c_{22} & a_{23}+c_{23} \\ a_{31}+c_{31} & a_{32}+c_{32} & a_{33}+c_{33} \end{bmatrix}$$

Example 1.1

Add the given matrices :

$$A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 0 & 8 \\ 9 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 & 5 \\ 7 & 9 & 0 \\ 8 & 1 & 4 \end{bmatrix}$$

Then:

$$A + B = \begin{bmatrix} 9 & 10 & 10 \\ 8 & 9 & 8 \\ 17 & 6 & 10 \end{bmatrix}$$

1.2.2 Subtraction Operation:

If there are two matrices, say A and B, are of same order then the sum of the two matrices can be written in form of "A+B " , and their corresponding elements can be written in form of

$d_{xy} = a_{xy} - c_{xy}$ for all the values of x and y

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Then;

$$A - C = \begin{bmatrix} a_{11} - c_{11} & a_{12} - c_{12} & a_{13} - c_{13} \\ a_{21} - c_{21} & a_{22} - c_{22} & a_{23} - c_{23} \\ a_{31} - c_{31} & a_{32} - c_{32} & a_{33} - c_{33} \end{bmatrix}$$

Example 1.2

Subtract the given matrices :

$$A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 0 & 8 \\ 9 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 & 5 \\ 7 & 9 & 0 \\ 8 & 1 & 4 \end{bmatrix}$$

Then:

$$A - B = \begin{bmatrix} -5 & 2 & 0 \\ 6 & 9 & -8 \\ -1 & -4 & -2 \end{bmatrix}$$

1.2.3 Scalar Multiplication:

In Scalar multiplication, a matrix of any order (say A) can be multiplied by any scalar number (say k). Then a new matrix is obtained whose each element is k times than the corresponding value of matrix A.

If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then,

$$C = kA = \begin{bmatrix} kc_{11} & kc_{12} & kc_{13} \\ kc_{21} & kc_{22} & kc_{23} \\ kc_{31} & kc_{32} & kc_{33} \end{bmatrix}$$

Example 1.3

Multiply the given matrix A by 3

$$A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 0 & 8 \\ 9 & 5 & 6 \end{bmatrix}$$

So; $B = 3A = \begin{bmatrix} 21 & 12 & 15 \\ 3 & 0 & 24 \\ 27 & 15 & 18 \end{bmatrix}$

1.2.4 Multiplication Operation:

If A and C are the two matrices, then the multiplication operation is only possible if the number of columns of matrix A is equal to the number of rows of matrix C.

If;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Then;

$$A \times C = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} + a_{13}c_{31} & a_{11}c_{12} + a_{12}c_{22} + a_{13}c_{32} \\ a_{21}c_{11} + a_{22}c_{21} + a_{23}c_{31} & a_{21}c_{12} + a_{22}c_{22} + a_{23}c_{32} \\ a_{31}c_{11} + a_{32}c_{21} + a_{33}c_{31} & a_{31}c_{12} + a_{32}c_{22} + a_{33}c_{32} \end{bmatrix}$$

In general, the multiplication operation can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1y} \\ a_{21} & a_{22} & . & . & a_{2y} \\ : & : & : & : & a_{2y} \\ : & : & : & : & : \\ a_{x1} & a_{x2} & . & . & a_{xy} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & . & . & c_{1y} \\ c_{21} & c_{22} & . & . & c_{2y} \\ : & : & : & : & c_{2y} \\ : & : & : & : & : \\ c_{x1} & c_{x2} & . & . & c_{xy} \end{bmatrix}$$

Then;

$$D = A \times C = \begin{bmatrix} d_{11} & d_{12} & . & . & d_{1y} \\ d_{21} & d_{22} & . & . & d_{2y} \\ : & : & : & : & d_{2y} \\ : & : & : & : & : \\ d_{x1} & d_{x2} & . & . & d_{xy} \end{bmatrix}$$

Example 1.4

Subtract the given matrices :

$$A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 0 & 8 \\ 9 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 6 & 5 \\ 7 & 9 & 0 \\ 8 & 1 & 4 \end{bmatrix}$$

Then:

$$A \times B = \begin{bmatrix} 65 & 33 & 88 \\ 58 & 28 & 107 \\ 93 & 52 & 72 \end{bmatrix}$$

1.3 TRANSPOSE OF MATRICES:

The matrix obtained by interchanging it rows and columns is known as Transpose of matrices.
So, if a matrix, say A, is of order $m \times n$ then it's transpose, say C, will be order $n \times m$.
And is denoted by A^T or A'.
If;

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 9 & 11 & 13 \\ 15 & 17 & 19 \end{bmatrix} \qquad \text{Then;} \quad B = A^T = \begin{bmatrix} 3 & 9 & 15 \\ 5 & 11 & 17 \\ 7 & 13 & 19 \end{bmatrix}$$

Example 1.5

Find the transpose of the given matrix:

$$A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 0 & 8 \\ 9 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & 1 & 9 \\ 4 & 0 & 5 \\ 5 & 8 & 6 \end{bmatrix}$$

1.4 TRACE OF MATRIX:

Sum of all elements in diagonal entries in a matrix is known as Trace of the matrix.
For E.g; If matrix A is of order $n \times n$, then

The trace of matrix is defined as $tr(A) = \sum_{i=1}^{n} a_{ii}$

1.5 LET US SUM UP:

In this chapter, we understood what matrices is and what types of operations can be performed using matrices. Later in upcoming chapters we will know how to find determinants and inverse of any matrices and many more.

1.6 GLOSSARY:

1. **Conformable** are the pairs of matrices where number of rows of one matrix is equal to number of columns of another matrix.
2. **Unit Matrix** is another name of Identity Matrix.
3. **Sparse Matrix** is a matrix whose most of the elements are zero.

1.7 ACTIVITY:

1. Zubin has two factories A and B. Each factory producing shoes for boys and girls in three different colours- Black, White, Red. The quantities produced by each factory is represented in matrices below. Let boys and girls be represented as columns 1 and 2, respectively, and the colours Black, White, Red can be represented as rows 1, 2, and 3 respectively. Find the total number of bags produced by both factories of each colour.

1.8 ASSIGNMENT:

1. Find the new matrix from the following matrices by applying Addition, Subtraction, and Multiplication Operations.

$$A = \begin{bmatrix} 5 & 8 & 6 & 7 \\ 6 & 7 & 1 & 4 \\ 9 & 4 & 2 & 0 \\ 2 & 1 & 9 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & 3 & 7 \\ 6 & 9 & 0 & 5 \\ 7 & 2 & 0 & 3 \\ 8 & 6 & 1 & 7 \end{bmatrix}$$

1.9 FURTHER READING:

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 2: DETERMINANTS

Unit Structure

2.0 INTRODUCTION:

Determinant is a number associated with the given square matrix. It is denoted with **|A|** or **det(A)** or $\triangle$**.** Further in this chapter we will learn many more operations and to find the inverse of a matrix.

2.1 MINORS OF DETERMINENTS:

Determinant obtained by removing the row and column to which the element of the original determinant belong is known as Minor of determinants.

If

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then

$$\text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Example 2.1

Find all minors of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$MINOR(A) \quad = \quad MINOR \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

$A$11=minor of 2= $\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}$ =5×2-6×1 =10-6=4

$A$12=minor of 3= $\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}$ =0×2-6×1=0-6=-6

$A$13=minor of 1= $\begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix}$ =0×1-5×1=0-5=-5

$A$21=minor of 0= $\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$ =3×2-1×1=6-1=5

$A22$=minor of 5= $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ =2×2-1×1=4-1=3

$A23$=minor of 6= $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ =2×1-3×1=2-3=-1

$A31$=minor of 1= $\begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix}$ =3×6-1×5=18-5=13

$A32$=minor of 1= $\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix}$ =2×6-1×0=12+0=12

$A33$=minor of 2= $\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$ =2×5-3×0=10+0=10

The minor matrix of A is [Aij]= $\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} 4 & -6 & -5 \\ 5 & 3 & -1 \\ 13 & 12 & 10 \end{vmatrix}$

2.2 CO-FACTORS OF DETERMINENTS:

Co-factor of any element of determinant is equal to corresponding minor with a proper sign attached to it. The general form is:

Co-factor of $a_{ij} = (-1)^{i+j} \left| \triangle_{ij} \right|$

Co-factor of a_{11} in 3×3 matrix = $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$= +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Example 2.2 :

Find the co-factor of the following matrix:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$COFACTOR(A) = COFACTOR \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Cofactor of 2=$A$11= + $\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}$ = +(5×2-6×1)=+(10-6)=4

Cofactor of 3=$A$12= - $\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}$ = -(0×2-6×1)=-(0-6)=6

Cofactor of 1=$A$13= + $\begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix}$ =+(0×1-5×1)=+(0-5)=-5

Cofactor of 0=$A$21= - $\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$ =-(3×2-1×1)=-(6-1)=-5

Cofactor of 5=$A$22= + $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ =+(2×2-1×1)=+(4-1)=3

Cofactor of 6=$A$23= - $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ =-(2×1-3×1)=-(2-3)=1

Cofactor of 1=$A$31= + $\begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix}$ =+(3×6-1×5)=+(18-5)=13

Cofactor of 1=$A$32= - $\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix}$ =-(2×6-1×0)=-(12+0)=-12

Cofactor of 2=$A$33= + $\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$ =+(2×5-3×0)=+(10+0)=10

The Cofactor matrix of A is [Aij]= $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & 6 & -5 \\ -5 & 3 & 1 \\ 13 & -12 & 10 \end{bmatrix}$

2.3 ADJOINT OF A MATRIX:

By replacing the element a_{ij} in the determinant |A| with its co-factor, the adjoint of a matrix A is formed. It is also known as the Adjoint matrix A. Adj(A) denotes it.

If; $|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then; $adj(A) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$

Example 2.3 :

Find the adjoint of the following matrix:

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution:

$Adj(A) =$ Adj $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (5\times 2-6\times 1) & -(0\times 2-6\times 1) & (0\times 1-5\times 1) \\ -(3\times 2-1\times 1) & (2\times 2-1\times 1) & -(2\times 1-3\times 1) \\ (3\times 6-1\times 5) & -(2\times 6-1\times 0) & (2\times 5-3\times 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} (10-6) & -(0-6) & (0-5) \\ -(6-1) & (4-1) & -(2-3) \\ (18-5) & -(12-0) & (10-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 6 & 5 \\ -5 & 3 & 1 \\ 13 & -12 & 10 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ 5 & 1 & 10 \end{bmatrix}^T$$

2.4 INVERSE OF A MATRIX:

Inverse of matrix exists only if,

$A.A^{-1} = I$; Where I is Identity Matrix.

And is denoted by A^{-1}. Most effective and easy way to find the Inverse of any matrix is Gauss-Jordan Elimination. And the method is described below.

For Instance, if we have to find the inverse of the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1y} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2y} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3y} \\ . & . & . & & & . \\ . & . & . & & & . \\ a_{x1} & a_{x2} & a_{x3} & . & . & a_{xy} \end{bmatrix}$$

We augment the identity matrix with the original matrix of the same order as the original matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & . & a_{1y} & : & 1 & 0 & 0 & . & 0 \\ a_{21} & a_{22} & a_{23} & . & a_{2y} & : & 0 & 1 & 0 & . & 0 \\ a_{31} & a_{32} & a_{33} & . & a_{3y} & : & 0 & 0 & 1 & . & 0 \\ . & . & . & & . & : & . & . & . & & . \\ . & . & . & & . & : & . & . & . & & . \\ a_{y1} & a_{y2} & a_{y3} & . & a_{yy} & : & 0 & 0 & 0 & . & 1 \end{bmatrix}$$

Gauss-Jordan elimination is performed using the first row as the pivot row of the augmented matrix of order nx(n+n).

First, divide the first row by a_{11}, then perform the following transformations

$R_x - a_{x1}R_1$ for x= 2,3,4,….,n

And hence we obtain a new matrix as follows:

$$\begin{bmatrix} 1 & a_{12} & a_{13} & . & a_{1y} & : & a_{1(y+1)} & 0 & 0 & . & 0 \\ 0 & a_{22} & a_{23} & . & a_{2y} & : & a_{2(y+1)} & 1 & 0 & . & 0 \\ 0 & a_{32} & a_{33} & . & a_{3y} & : & a_{3(y+1)} & 0 & 1 & . & 0 \\ . & . & . & & & : & . & . & . & & . \\ . & . & . & & . & : & . & . & . & & . \\ 0 & a_{y2} & a_{y3} & . & a_{yy} & : & a_{y(y+1)} & 0 & 0 & . & 1 \end{bmatrix}$$

Next, use the second row as a pivot row. After dividing the second row by a22, perform the following transformations

$R_i - a_{iy}R_y$ for i=1,2,3,4,….,(y-1)

And we obtain a new matrix:

$$\begin{bmatrix} 1 & 0 & a_{13} & . & a_{1y} & : & a_{1(y+1)} & a_{1(y+2)} & 0 & . & 0 \\ 0 & 1 & a_{23} & . & a_{2y} & : & a_{2(y+1)} & a_{2(y+2)} & 0 & . & 0 \\ 0 & 0 & a_{33} & . & a_{3y} & : & a_{3(y+1)} & a_{3(y+2)} & 1 & . & 0 \\ . & . & . & & & : & . & . & . & & . \\ . & . & . & & . & : & . & . & . & & . \\ 0 & 0 & a_{y3} & . & a_{yy} & : & a_{y(y+1)} & a_{y(y+1)} & 0 & . & 1 \end{bmatrix}$$

Similiarly this process is continued till nth row is used as pivot row.
And we get the a matrix as follows:

$$\begin{bmatrix} 1 & 0 & 0 & . & 0 & : & a_{1(y+1)} & a_{1(y+2)} & a_{1(y+3)} & . & a_{1(y+y)} \\ 0 & 1 & 0 & . & 0 & : & a_{2(y+1)} & a_{2(y+2)} & a_{2(y+3)} & . & a_{2(y+y)} \\ 0 & 0 & 1 & . & 0 & : & a_{3(y+1)} & a_{3(y+2)} & a_{3(y+3)} & . & a_{3(y+y)} \\ . & . & . & & & : & . & . & . & & . \\ . & . & . & & . & : & . & . & . & & . \\ 0 & 0 & 0 & . & 1 & : & a_{y(y+1)} & a_{y(y+2)} & a_{y(y+3)} & . & a_{y(y+y)} \end{bmatrix}$$

2.5 GLOSSARY:

1. **Adjoint Matrix** is the transpose of the respective co-factor matrix for the given matrix.
2. If a matrix is in row-echelon form, then the first nonzero entry of each row is called a pivot.

2.6 ACTIVITY:

1. Find $(A \times B)' = B' \times A'$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

2. Find $(A \times B)' = B' \times A'$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

2.7 ASSIGNMENT:

1. Find A^{-1}

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

2. Find A^{-1}

$$A = \begin{bmatrix} 7 & 5 & 0 \\ 6 & 8 & 4 \\ 4 & 2 & 1 \end{bmatrix}$$

2.8 FURTHER READING:

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012.

UNIT 3: SOLVING SIMULTANEOUS LINEAR ALGEBRAIC EQUATION

Unit Structure

3.0 INTRODUTION

A number of problems involving algebraic equations appear in various fields of study. Due to the fact that these equations are often linear, we will focus on methods that can be used to solve systems of linear equations throughout this chapter.

Algebraic systems of linear equations are usually solved using two common methods:

1. Through the combination of equations that eliminate the variables.
2. Using Cramer's rule, a method that uses determinants.

There appears to be an advantage to Cramer's rule over the elimination method when there are fewer equations to solve. It is however impossible to solve large numbers of equations using Cramer's rule, which is a method of determining determinants with minor expansion. Using Cramer's rule, Forsythe proved that evaluating determinants with expansion by minors requires multiplications in order to solve a system of n linear equations.

Thus, even the solution of 10 equations would requires 35, 92, 51,200 multiplications. Obviously, a solution involving the use of determinants with expansion by minors is not practical. Even if determinants are evaluated by the use of Gauss elimination method, the use of Cramer's rule is very inefficient.

A system of linear equations involving n-equations and n unknowns will be of general form as follows:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad &\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n &= b_n \quad \dots\dots\dots(1)
\end{aligned}$$

In nonhomogeneous linear equation systems, all b's cannot equal zero. Independent linear equation systems have a unique solution, while nonhomogeneous systems do not.

In nonhomogeneous linear equation systems, all b's cannot equal zero. Independent linear equation systems have a unique solution, while

nonhomogeneous systems do not. The system of linear equations is called homogeneous if all the b's are zero. If the equations are not independent, there is a nontrivial solution (other than x1=x2=x3=, x=0).

3.1 SOLUTION OF NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

There are two types of method to obtain solution of non-homogeneous system of linear equations. These are:

1. Direct Method
2. \ Iterative Method

3.2 DIRECT METHOD

Amongst direct methods, we will discuss:

1. Gauss-Elimination Method
2. Gauss Jordan Method
3. Matrix-Inversion Method

3.2.1 Gauss-Elimination Method

In this method, the system of equations is reduced to an equivalent upper-triangular system, which can be solved by back substitution.

The system of n linear equations in n unknowns can be given by:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Step1: An upper-triangular system is obtained by eliminating the unknowns. In order to eliminate x1 from the second equation, multiply the first equation by $(-a_{21}/a_{11})$ and we get

$$-a_{21}x_1 - a_{12}\frac{a_{21}}{a_{11}}x_2 - a_{13}\frac{a_{21}}{a_{11}}x_3 - ... - a_{1n}\frac{a_{21}}{a_{11}}x_n = -b_1\frac{a_{21}}{a_{11}}$$

On Adding above equation with the second equation of 4.1 we get

$$(a_{22} - a_{12}\frac{a_{21}}{a_{11}})x_2 + (a_{23} - a_{13}\frac{a_{21}}{a_{11}})x_3 + ... + (a_{2n} - a_{1n}\frac{a_{21}}{a_{11}})x_n = b_2 - b_1\frac{a_{21}}{a_{11}}$$

It can also be written as

$$a'_{32}x_2 + a'_{33}x_3 + ... + a'_{3n}x_n = b'_3$$

Where $a'_{22}=a_{22} - a_{12}(a_{21}/a_{11})$, etc. As a result, primes indicate that the original element has changed in value. In the same way, we can multiply the first equation by $-a_{31}/a_{11}$ and add it to the third equation of 4.1. This eliminates the unknown x1 from the third equation of 4.1 and we obtain

$$a'_{32}x_2 + a'_{33}x_3 + ... + a'_{3n}x_n = b'_3$$

Similarly, we can eliminate x1 from the remaining equations, and after removing x1 from eq 4.1, we obtain the system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a'_{22}x_2 + a'_{23}x_3 + ... + a'_{2n}x_n = b'_2$$

$$a'_{32}x_2 + a'_{33}x_3 + ... + a'_{3n}x_n = b'_3$$

.

.

.

$$a'_{n2}x_2 + a'_{n3}x_3 + ... + a'_{nn}x_n = b'_n \qquad (4.2)$$

Next, we eliminate x from the last (n-2) equations in Eq (4.2). In order to obtain the above system, we have multiplied the first row by (-an/a), i.e. divided it by a which is assumed to be non-zero. As a result, the first equation in the system (4.2) is called the pivot equation, and a is called the pivotal element. If a=0, the method fails. This important point will be discussed after the elimination method has been described. We multiply the second equation by (-a'32/a'22) and add it to the third equation to eliminate x2 from Eq (4.2). If we repeat this process with the remaining equations, we obtain the system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a'_{22}\,x_2 + a'_{23}\,x_3 + ... + a'_{2n}\,x_n = b'_2$$

$$a''_{33}\,x_3 + ... + a''_{3n}\,x_n = b'_3$$

.

.

$$a''_{n3}\,x_3 + ... + a''_{nn}\,x_n = b'_n \qquad (4.3)$$

The 'double primes' in Eq(4.3) indicate that the elements have changed twice. This procedure can be repeated to eliminate x3 from the fourth equation onwards, x4 from the fifth equation onwards, etc., till we finally obtain the upper-triangular form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a'_{22}\,x_2 + a'_{23}\,x_3 + ... + a'_{2n}\,x_n = b'_2$$

$$a''_{33}\,x_3 + ... + a''_{3n}\,x_n = b'_3$$

.

.

$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)} \qquad (4.4)$$

$a_{nn}^{(n-1)}$ indicates that the element a_{nn}'s value has changed (n-1) times. Thus, we have eliminated the unknowns and reduced to the upper-triangular form.

Step 2: From the last equation of the system of the Eq4.4, we obtain

$$a_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}} \tag{4.5}$$

3.2.2 Gauss Jordan Method

This method is a modification of Gauss elimination, except it does require for back substitution to obtain the solution. When an unknown is eliminated, it is eliminated from all equations.

Example 3.1

Solve equations 2x + y + z = 5,3x + 5y + 2z = 15, 2x + y + 4z =8 using Gauss-Jordan elimination method

Solution:

Total Equations are 3

$2x+y+z=5 \rightarrow (1)$
$3x+5y+2z=15 \rightarrow (2)$
$2x+y+4z=8 \rightarrow (3)$

Converting given equations into matrix form

$$\begin{matrix} 2 & 1 & 1 & 5 \\ 3 & 5 & 2 & 15 \\ 2 & 1 & 4 & 8 \end{matrix}$$

$R1 \leftarrow R1 \div 2$

$$= \begin{matrix} 1 & 0.5 & 0.5 & 2.5 \\ 3 & 5 & 2 & 15 \\ 2 & 1 & 4 & 8 \end{matrix}$$

$R2 \leftarrow R2 - 3 \times R1$

$$= \begin{matrix} 1 & 0.5 & 0.5 & 2.5 \\ 0 & 3.5 & 0.5 & 7.5 \\ 2 & 1 & 4 & 8 \end{matrix}$$

$R3 \leftarrow R3 - 2 \times R1$

$$= \begin{matrix} 1 & 0.5 & 0.5 & 2.5 \\ 0 & 3.5 & 0.5 & 7.5 \\ 0 & 0 & 3 & 3 \end{matrix}$$

$R2 \leftarrow R2 \times 0.2857$

$$= \begin{matrix} 1 & 0.5 & 0.5 & 2.5 \\ 0 & 1 & 0.1429 & 2.1429 \\ 0 & 0 & 3 & 3 \end{matrix}$$

$R1 \leftarrow R1 - 0.5 \times R2$

$$= \begin{matrix} 1 & 0 & 0.4286 & 1.4286 \\ 0 & 1 & 0.1429 & 2.1429 \\ 0 & 0 & 3 & 3 \end{matrix}$$

$R3 \leftarrow R3 \div 3$

$$= \begin{matrix} 1 & 0 & 0.4286 & 1.4286 \\ 0 & 1 & 0.1429 & 2.1429 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$R1 \leftarrow R1 - 0.4286 \times R3$

$$= \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0.1429 & 2.1429 \\ 0 & 0 & 1 & 1 \end{matrix}$$

$R2 \leftarrow R2 - 0.1429 \times R3$

$$= \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{matrix}$$

i.e.
$x=1$
$y=2$
$z=1$

Solution By Gauss Jordan elimination method $x=1, y=2$ and $z=1$

Example 3.2

Solve equations 2x + 5y =16, 3x + y =11 using Gauss-Jordan Elimination method

Solution:
Total equations are 2

$2x+5y=16 \rightarrow (1)$
$3x+y=11 \rightarrow (2)$

Converting given equations into matrix form

$$\begin{matrix} 2 & 5 & 16 \\ 3 & 1 & 11 \end{matrix}$$

$R1 \leftarrow R1 \div 2$

$$= \begin{matrix} 1 & 2.5 & 8 \\ 3 & 1 & 11 \end{matrix}$$

$R2 \leftarrow R2 - 3 \times R1$

$$= \begin{matrix} 1 & 2.5 & 8 \\ 0 & -6.5 & -13 \end{matrix}$$

$R2 \leftarrow R2 \times -0.1538$

$$= \begin{matrix} 1 & 2.5 & 8 \\ 0 & 1 & 2 \end{matrix}$$

$R1 \leftarrow R1 - 2.5 \times R2$

$$= \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{matrix}$$

i.e.
$x=3$
$y=2$

Solution By Gauss Jordan elimination method $x=3$and $y=2$.

Example 3.3

Solve Equations 2x+5y=21, x+2y=8 using Gauss Elimination Back Substitution method.

Solution:
Total Equations are 2

$2x+5y=21 \rightarrow (1)$

$x+2y=8 \rightarrow (2)$

Converting given equations into matrix form

$$\begin{matrix} 2 & 5 & 21 \\ 1 & 2 & 8 \end{matrix}$$

$R2 \leftarrow R2 - 0.5 \times R1$

$$= \begin{matrix} 2 & 5 & 21 \\ 0 & -0.5 & -2.5 \end{matrix}$$

i.e.

$2x+5y=21 \rightarrow (1)$

$-0.5y=-2.5 \rightarrow (2)$

Now use back substitution method
From (2)
$-0.5y=-2.5$

$\Rightarrow y=-2.5\times-2=5$

From (1)
$2x+5y=21$

$\Rightarrow 2x+5(5)=21$

$\Rightarrow 2x+25=21$

$\Rightarrow 2x=21-25$

$\Rightarrow 2x=-4$

$\Rightarrow x=-42=-2$

Solution using back substitution method.
$x=-2$ and $y=5$

Example 3.4

Solve Equations 2x+y=8, x+2y=1 using Gauss Elimination Back Substitution method

Solution:
Total Equations are 2

$2x+y=8\rightarrow(1)$
$x+2y=1\rightarrow(2)$

Converting given equations into matrix form

2	1	8
1	2	1

$R2\leftarrow R2-0.5\times R1$

=

2	1	8
0	1.5	-3

i.e.

$2x+y=8 \rightarrow (1)$
$1.5y=-3 \rightarrow (2)$

Now use back substitution method
From (2)
$1.5y=-3$
$\Rightarrow y=-3 \times 0.6667=-2$

From (1)
$2x+y=8$
$\Rightarrow 2x+(-2)=8$
$\Rightarrow 2x-2=8$
$\Rightarrow 2x=8+2$
$\Rightarrow 2x=10$
$\Rightarrow x=102=5$

Solution using back substitution method.
$x = 5$ and $y = -2$

3.2.3 Matrix Inversion Method

This method can be applied only when the coefficient matrix is a square matrix and non-singular.

Consider the matrix equation

$$AX = B, \quad \ldots (1)$$

where A is a square matrix and non-singular. Since A is non-singular, A^{-1} exists and $A^{-1}A = AA^{-1} = I$. Pre-multiplying both sides of (1) by A^{-1}, we get $A^{-1}(AX) = A^{-1}B$. That is, $(A^{-1}A)X = A^{-1}B$. Hence, we get $X = A^{-1}B$.

Example 3.5

Solve Equations 2x+3y-z=5,3x+2y+z=10,x-5y+3z=0 using Inverse Matrix method

Solution:
Here 2x+3y-z=5
3x+2y+z=10
x-5y+3z=0

Now converting given equations into matrix form
[[2,3,-1],[3,2,1],[1,-5,3]] [[x],[y],[z]]=[[5],[10],[0]]

Now, A = [[2,3,-1],[3,2,1],[1,-5,3]], X = [[x],[y],[z]] and B = [[5],[10],[0]]

$\therefore AX = B$

$\therefore X = A^{-1} B$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 2 & 1 \\ -5 & 3 \end{vmatrix} -3 \times \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} -1 \times \begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix}$$

$=2 \times (2 \times 3 - 1 \times (-5)) -3 \times (3 \times 3 - 1 \times 1) -1 \times (3 \times (-5) - 2 \times 1)$
$=2 \times (6 +5) -3 \times (9 -1) -1 \times (-15 -2)$
$=2 \times (11) -3 \times (8) -1 \times (-17)$
$= 22 -24 +17$
$=15$

Here, $|A| = 15 \neq 0$

$\therefore A^{-1}$ is possible.

$$\text{Adj}(A) = \text{Adj} \begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} 2 & 1 \\ -5 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -5 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(2 \times 3 - 1 \times (-5)) & -(3 \times 3 - 1 \times 1) & +(3 \times (-5) - 2 \times 1) \\ -(3 \times 3 - (-1) \times (-5)) & +(2 \times 3 - (-1) \times 1) & -(2 \times (-5) - 3 \times 1) \\ +(3 \times 1 - (-1) \times 2) & -(2 \times 1 - (-1) \times 3) & +(2 \times 2 - 3 \times 3) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(6+5) & -(9-1) & +(-15-2) \\ -(9-5) & +(6+1) & -(-10-3) \\ +(3+2) & -(2+3) & +(4-9) \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & -8 & -17 \\ -4 & 7 & 13 \\ 5 & -5 & -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & -4 & 5 \\ -8 & 7 & -5 \\ -17 & 13 & -5 \end{bmatrix}$$

"Now, "A^(-1)=1/|A| × Adj(A)

"Here, "X = A^(-1) × B

:. X = 1/|A| × Adj(A) × B

$$= 1/(15) \times \begin{bmatrix} 11 & -4 & 5 \\ -8 & 7 & -5 \\ -17 & 13 & -5 \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$= 0.0667 \times \begin{bmatrix} 11\times5-4\times10+5\times0 \\ -8\times5+7\times10-5\times0 \\ -17\times5+13\times10-5\times0 \end{bmatrix}$$

$$= 0.0667 \times \begin{bmatrix} 15 \\ 30 \end{bmatrix}$$

45

$$= \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

:.[[x],[y],[z]]=[[1],[2],[3]]

:.x=1,y=2,z=3

Example 3.6

Solve Equations 4x+5y+z=9,x-y-2z=7,x-y+z=13 using Inverse Matrix method

Solution:
Here $4x+5y+z=9$
$x-y-2z=7$
$x-y+z=13$

Now converting given equations into matrix form
[4511-1-21-11][*xyz*]=[9713]

Now, A = [4511-1-21-11], X = [*xyz*] and B = [9713]

$\therefore AX=B$

$\therefore X=A$-1B

$$|A| = \begin{matrix} 4 & 5 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{matrix}$$

$$= 4 \times \begin{matrix} -1 & -2 \\ -1 & 1 \end{matrix} -5 \times \begin{matrix} 1 & -2 \\ 1 & 1 \end{matrix} +1 \times \begin{matrix} 1 & -1 \\ 1 & -1 \end{matrix}$$

=4×(-1×1-(-2)×(-1))-5×(1×1-(-2)×1)+1×(1×(-1)-(-1)×1)

=4×(-1-2)-5×(1+2)+1×(-1+1)

$=4\times(-3)-5\times(3)+1\times(0)$

$=-12-15+0$

$=-27$

Here, $|A|=-27\neq0$

$\therefore A^{-1}$ is possible.

$$Adj(A) = \text{Adj}\begin{vmatrix} 4 & 5 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} +\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 5 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 5 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} & +\begin{vmatrix} 4 & 5 \\ 1 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1\times1-(-2)\times(-1)) & -(1\times1-(-2)\times1) & +(1\times(-1)-(-1)\times1) \\ -(5\times1-1\times(-1)) & +(4\times1-1\times1) & -(4\times(-1)-5\times1) \\ +(5\times(-2)-1\times(-1)) & -(4\times(-2)-1\times1) & +(4\times(-1)-5\times1) \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1-2) & -(1+2) & +(-1+1) \\ -(5+1) & +(4-1) & -(-4-5) \\ +(-10+1) & -(-8-1) & +(-4-5) \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & -3 & 0 \\ -6 & 3 & 9 \end{bmatrix}^T$$

-9 9 -9

$$= \begin{bmatrix} -3 & -6 & -9 \\ -3 & 3 & 9 \\ 0 & 9 & -9 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} \times Adj(A)$

Here, $X = A^{-1} \times B$

$\therefore X = \frac{1}{|A|} \times Adj(A) \times B$

$$= \frac{1}{-27} \times \begin{bmatrix} -3 & -6 & -9 \\ -3 & 3 & 9 \\ 0 & 9 & -9 \end{bmatrix} \times \begin{bmatrix} 9 \\ 7 \\ 13 \end{bmatrix}$$

$$= -0.037 \times \begin{bmatrix} -3 \times 9 - 6 \times 7 - 9 \times 13 \\ -3 \times 9 + 3 \times 7 + 9 \times 13 \\ 0 \times 9 + 9 \times 7 - 9 \times 13 \end{bmatrix}$$

$$= -0.037 \times \begin{bmatrix} -186 \\ 111 \\ -54 \end{bmatrix}$$

$$= \begin{bmatrix} 6.8889 \\ -4.1111 \\ 2 \end{bmatrix}$$

$\therefore [x, y, z] = [6.8889 \; -4.1111 \; 2]$

$\therefore x = 62/9$, $y = -37/9$, $z = 2$

Check Your Progress-1

1. Solve the following system of linear equations

$9x_1 + 12x_2 + 27x_3 = 33$

$5x_1 + 7x_2 - 3x_3 = 23$

$11x_1 - 21x_2 + 2x_3 = 57$

Using Gauss-Elimination Method

2. Solve the following system of linear equations

$2x_1 - 2x_2 + 5x_3 = 13$

$2x_1 + 3x_2 + 4x_3 = 20$

$x_1 + x_2 - 3x_3 = 19$

Using Gauss-Jordan Method / Matrix Inversion Method

3.3 ITERATIVE METHODS

As a numerical technique, Gaussian elimination is rather unusual because it is direct. That is, a solution is obtained after a single application of Gaussian elimination. Once a "solution" has been obtained, Gaussian elimination offers no method of refinement. The lack of refinements can be a problem because, as the previous section shows, Gaussian elimination is sensitive to rounding error. Numerical techniques more commonly involve an iterative method. For example, in calculus you probably studied Newton's iterative method for approximating the zeros of a differentiable function. In this section you will look at two iterative methods for approximating the solution of a system of n linear equations in n variables.

1. Jacobi's Method
2. Gauss Seidel Method

3.3.1 Jacobi's Method

The first iterative technique is called the Jacobi method, after Carl Gustav Jacob Jacobi (1804–1851). This method makes two assumptions: the system given by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n = b_3$$

$$\begin{array}{ccccc} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

has a unique solution and the coefficient matrix *A* has no zeros on its main diagonal.
If any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

To begin the Jacobi method, solve the first equation for the second equation for and so on, as follows.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n)$$

.

.

.

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n3}x_3 - \ldots - a_{nn}x_{n-1})$$

Then make an initial approximation of the solution,

$$(x_1, x_2, x_3, \ldots, x_n)$$

And the substitute these values of x_i into the right-hand side of the rewritten equations to obtain the first approximation. After this procedure has been completed, one iteration has been performed. In the same way, the second approximation is formed by substituting the first approximation's *x*-values into the right-hand side of the rewritten equations. By repeated iterations, you will form a sequence of approximations that often converges to the actual solution.

Example 3.7

Solve Equations 2x-y+3z=1,-3x+4y-5z=0,x+3y-6z=0 using Gauss Jacobi method

Solution:

Total Equations are 3

$2x-y+3z=1$

$-3x+4y-5z=0$

$x+3y-6z=0$

From the above equations

x_{k+1} = 12(1+y_k-3z_k)
y_{k+1} = 1/4(0+3x_k+5z_k)
z_{k+1} = 1/-6(0-x_k-3y_k)

Initial gauss (x,y,z)=(0,0,0)

Solution steps are
1*st* Approximation
x_1 = 1/2[1+(0)-3(0)] = 1/2[1] = 0.5
y_1 = 1/4[0+3(0)+5(0)] = 1/4[0] = 0
z_1 = 1/-6[0-(0)-3(0)] = 1/-6[0] = 0

2*nd* Approximation
x_2 = 1/2[1+(0)-3(0)] = 1/2[1] = 0.5
y_2 = 1/4[0+3(0.5)+5(0)] = 1/4[1.5] = 0.375
z_2 = 1/-6[0-(0.5)-3(0)] = 1/-6[-0.5] = 0.0833

3*rd* Approximation
x_3 = 1/2[1+(0.375)-3(0.0833)] = 1/2[1.125] = 0.5625
y_3 = 1/4[0+3(0.5)+5(0.0833)] = 1/4[1.9167] = 0.4792
z_3 = 1/-6[0-(0.5)-3(0.375)] = 1/-6[-1.625] = 0.2708

4*th* Approximation
x_4 = 1/2[1+(0.4792)-3(0.2708)] = 1/2[0.6667] = 0.3333
y_4 = 1/4[0+3(0.5625)+5(0.2708)] = 1/4[3.0417] = 0.7604
z_4 = 1/-6[0-(0.5625)-3(0.4792)] = 1/-6[-2] = 0.3333

5*th* Approximation
x_5 = 1/2[1+(0.7604)-3(0.3333)] = 1/2[0.7604] = 0.3802
y_5 = 1/4[0+3(0.3333)+5(0.3333)] = 1/4[2.6667] = 0.6667
z_5 = 1/-6[0-(0.3333)-3(0.7604)] = 1/-6[-2.6146] = 0.4358

6*th* Approximation
x_6 = 1/2[1+(0.6667)-3(0.4358)] = 1/2[0.3594] = 0.1797
y_6 = 1/4[0+3(0.3802)+5(0.4358)] = 1/4[3.3194] = 0.8299
z_6 = 1/-6[0-(0.3802)-3(0.6667)] = 1/-6[-2.3802] = 0.3967

Example 3.8

Solve Equations 2x+y+3z=7, 5x+y+2z=13, x+3y+z=19 using Gauss Jacobi method

Solution:

Total Equations are 3

$2x+y+3z=7$

$5x+y+2z=13$

$x+3y+z=19$

The coefficient matrix of the given system is not diagonally dominant.
Hence, we re-arrange the equations as follows, such that the elements in the coefficient matrix are diagonally dominant.

$5x+y+2z=13$

$x+3y+z=19$

$2x+y+3z=7$

From the above equations

$x_{k+1} = 1/5(13-y_k-2z_k)$

$y_{k+1} = 1/3(19-x_k-z_k)$

$z_{k+1} = 1/3(7-2x_k-y_k)$

Initial gauss (x, y, z)=(0,0,0)

Solution steps are

1*st* Approximation

$x_1 = 1/5\ [13-(0)-2(0)] = 1/5[13] = 2.6$

$y_1 = 1/3[19-(0)-(0)] = 1/3[19] = 6.3333$

$z_1 = 1/3[7-2(0)-(0)] = 1/3[7] = 2.3333$

2*nd* Approximation

$x_2 = 1/5[13-(6.3333)-2(2.3333)] = 1/5[2] = 0.4$

$y_2 = 1/3[19-(2.6)-(2.3333)] = 1/3[14.0667] = 4.6889$

$z_2 = 1/3[7-2(2.6)-(6.3333)] = 1/3[-4.5333] = -1.5111$

3*rd* Approximation

$x_3 = 1/5[13-(4.6889)-2(-1.5111)] = 1/5[11.3333] = 2.2667$

$y_3 = 1/3[19-(0.4)-(-1.5111)] = 1/3[20.1111] = 6.7037$

$z3=1/3[7-2(0.4)-(4.6889)] = 1/3[1.5111] = 0.5037$

4*th* Approximation
x_4 = 1/5[13-(6.7037)-2(0.5037)] = 1/5[5.2889] = 1.0578
y_4 = 1/3[19-(2.2667)-(0.5037)] = 1/3[16.2296] = 5.4099
z_4 = 1/3[7-2(2.2667)-(6.7037)] = 1/3[-4.237] = -1.4123

5*th* Approximation
x_5 = 1/5[13-(5.4099)-2(-1.4123)] = 1/5[10.4148] = 2.083
y_5 = 1/3[19-(1.0578)-(-1.4123)] = 1/3[19.3546] = 6.4515
z_5 = 1/3[7-2(1.0578)-(5.4099)] = 1/3[-0.5254] = -0.1751

6*th* Approximation
x_6 = 1/5[13-(6.4515)-2(-0.1751)] = 1/5[6.8988] = 1.3798
y_6 = 1/3[19-(2.083)-(-0.1751)] = 1/3[17.0922] = 5.6974
z_6 = 1/3[7-2(2.083)-(6.4515)] = 1/3[-3.6174] = -1.2058

3.3.2 The Gauss-Seidel Method

Gauss-Seidel is a modification of the Jacobi method named after Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). For the same degree of accuracy, this modification is no more difficult to use than the Jacobi method.

The Jacobi method preserves the values of xi obtained in the nth approximation. It remains unchanged until the entire $(n+1)^{th}$ approximation has been calculated. The Gauss-Seidel method, however, uses the new values of xi as soon as they are known. The new x1 and x2 are then used in the third equation to obtain the new x3, and so on, once x1 is determined from the first equation.

Example 3.9

Solve Equations 5x + 3y -2z=11, -3x + 4y- 5z=9, x + 3y- 6z = 0 using Gauss Seidel method

Solution:
Total Equations are 3

$5x+3y-2z=11$
$-3x+4y-5z=9$
$x+3y-6z=0$

From the above equations
$x_{k+1} = 1/5(11-3y_k+2z_k)$
$y_{k+1} = 1/4(9+3x_{k+1}+5z_k)$
$z_{k+1} = 1/-6(0-x_{k+1}-3y_{k+1})$

Initial gauss (x,y,z)=(0,0,0)

Solution steps are

1*st* Approximation
x_1 = 1/5[11-3(0)+2(0)] = 1/5[11] = 2.2
y_1 = 1/4[9+3(2.2)+5(0)] = 1/4[15.6] = 3.9
z_1 = 1/-6[0-(2.2)-3(3.9)] = 1/-6[-13.9] = 2.3167

2*nd* Approximation
x_2 = 1/5[11-3(3.9)+2(2.3167)] = 1/5[3.9333] = 0.7867
y_2 = 1/4[9+3(0.7867)+5(2.3167)] = 1/4[22.9433] = 5.7358
z_2 = 1/-6[0-(0.7867)-3(5.7358)] = 1/-6[-17.9942] = 2.999

3*rd* Approximation
x_3 = 1/5[11-3(5.7358)+2(2.999)] = 1/5[-0.2094] = -0.0419
y_3 = 1/4[9+3(-0.0419)+5(2.999)] = 1/4[23.8695] = 5.9674
z_3 = 1/-6[0-(-0.0419)-3(5.9674)] = 1/-6[-17.8602] = 2.9767

4*th* Approximation
x_4 = 1/5[11-3(5.9674)+2(2.9767)] = 1/5[-0.9487] = -0.1897
y_4 = 1/4[9+3(-0.1897)+5(2.9767)] = 1/4[23.3143] = 5.8286
z_4 = 1/-6[0-(-0.1897)-3(5.8286)] = 1/-6[-17.296] = 2.8827

5*th* Approximation
x_5 = 1/5[11-3(5.8286)+2(2.8827)] = 1/5[-0.7204] = -0.1441
y_5 = 1/4[9+3(-0.1441)+5(2.8827)] = 1/4[22.9811] = 5.7453
z_5 = 1/-6[0-(-0.1441)-3(5.7453)] = 1/-6[-17.0917] = 2.8486

6*th* Approximation
x_6 = 1/5[11-3(5.7453)+2(2.8486)] = 1/5[-0.5386] = -0.1077
y_6 = 1/4[9+3(-0.1077)+5(2.8486)] = 1/4[22.92] = 5.73
z_6 = 1/-6[0-(-0.1077)-3(5.73)] = 1/-6[-17.0823] = 2.847

7*th* Approximation
x_7 = 1/5[11-3(5.73)+2(2.847)] = 1/5[-0.4959] = -0.0992
y_7 = 1/4[9+3(-0.0992)+5(2.847)] = 1/4[22.9377] = 5.7344
z_7 = 1/-6[0-(-0.0992)-3(5.7344)]=1/-6[-17.1041] = 2.8507

8*th* Approximation
x_8 = 1/5[11-3(5.7344)+2(2.8507)] = 1/5[-0.5019] = -0.1004
y_8 = 1/4[9+3(-0.1004)+5(2.8507)] = 1/4[22.9523] = 5.7381
z_8 = 1/-6[0-(-0.1004)-3(5.7381)] = 1/-6[-17.1138] = 2.8523

9*th* Approximation

x_9 = 1/5[11-3(5.7381)+2(2.8523)] = 1/5[-0.5096] = -0.1019
y_9 = 1/4[9+3(-0.1019)+5(2.8523)] = 1/4[22.9558] = 5.7389
z_9 = 1/-6[0-(-0.1019)-3(5.7389)] = 1/-6[-17.1149] = 2.8525

10*th* Approximation
x_{10} = 1/5[11-3(5.7389)+2(2.8525)] = 1/5[-0.5119] = -0.1024
y_{10} = 1/4[9+3(-0.1024)+5(2.8525)] = 1/4[22.9553] = 5.7388
z_{10} = 1/-6[0-(-0.1024)-3(5.7388)] = 1/-6[-17.1141] = 2.8524

Solution By Gauss Seidel Method.
x = -0.1024 ≅ -0.1
y = 5.7388 ≅ 5.74
z = 2.8524 ≅ 2.85

Example 3.10

Solve Equations x+7y-3z=1, 3x-7y-z=7, 6x+2y-6z=8 using Gauss Seidel method

Solution:
Total Equations are 3

$x+7y-3z=1$
$3x-7y-z=7$
$6x+2y-6z=8$

From the above equations
$x_{k+1} = 1/1(1-7y_k+3z_k)$
$y_{k+1}=1/-7(7-3x_{k+1}+zk)$
$z_{k+1} = 1/-6(8-6x_{k+1}-2y_{k+1})$

Initial gauss (x,y,z)=(0,0,0)

Solution steps are
1*st* Approximation
x_1 = 1/1[1-7(0)+3(0)] = 1/1[1] = 1
y_1 = 1/-7[7-3(1)+(0)] = 1/-7[4] = -0.5714
z_1 = 1/-6[8-6(1)-2(-0.5714)] = 1/-6[3.1429] = -0.5238

2*nd* Approximation
x_2 = 1/1[1-7(-0.5714)+3(-0.5238)] = 1/1[3.4286] = 3.4286
y_2 = 1/-7[7-3(3.4286)+(-0.5238)] = 1/-7[-3.8095] = 0.5442

z_2 = 1/-6[8-6(3.4286)-2(0.5442)] = 1/-6[-13.6599] = 2.2766

3*rd* Approximation

x_3 = 1/1[1-7(0.5442)+3(2.2766)] = 1/1[4.0204] = 4.0204

y_3 = 1/-7[7-3(4.0204)+(2.2766)] = 1/-7[-2.7846] = 0.3978

z_3 = 1/-6[8-6(4.0204)-2(0.3978)] = 1/-6[-16.918] = 2.8197

4*th* Approximation

x_4 = 1/1[1-7(0.3978)+3(2.8197)] = 1/1[6.6744] = 6.6744

y_4 = 1/-7[7-3(6.6744)+(2.8197)] = 1/-7[-10.2036] = 1.4577

z_4 = 1/-6[8-6(6.6744)-2(1.4577)] = 1/-6[-34.962] = 5.827

5*th* Approximation

x_5 = 1/1[1-7(1.4577)+3(5.827)] = 1/1[8.2773] = 8.2773

y_5 = 1/-7[7-3(8.2773)+(5.827)] = 1/-7[-12.005] = 1.715

z_5 = 1/-6[8-6(8.2773)-2(1.715)] = 1/-6[-45.094] = 7.5157

6*th* Approximation

x_6 = 1/1[1-7(1.715)+3(7.5157)] = 1/1[11.542] = 11.542

y_6 = 1/-7[7-3(11.542)+(7.5157)] = 1/-7[-20.1103] = 2.8729

z_6 = 1/-6[8-6(11.542)-2(2.8729)] = 1/-6[-66.9978] = 11.1663

7*th* Approximation

*x*7=11[1-7(2.8729)+3(11.1663)]=11[14.3886]=14.3886

y_7 = 1/-7[7-3(14.3886)+(11.1663)]=1/-7[-24.9994] = 3.5713

z_7 = 1/-6[8-6(14.3886)-2(3.5713)] = 1/-6[-85.4741] = 14.2457

Check Your Progress-2

1. Solve the following system of equations

   $6x_1 + 2x_2 – x_3 = 1$

   $7x_1 + 2x_2 – 4x_3 = -8$

   $x_1 – 3x_2 - 9x_3 = 27$

2. Solve the following system of equations

   $x_1 + 7x_2 + 6x_3 = 65$

   $-2x_1 + 25x_2 + x_3 = 46$

   $x_1 + 3x_2 + 5x_3 =34$

3.4 LET US SUM UP

In this unit, we:

- We studied different methods to solved linear equations

3.5 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

4. $x_1 = 4.7752$, $x_2 = -0.2381$, $x_3 = -0.2637$
5. $x_1 = 12.2745$, $x_2 = 1.0196$, $x_3 = -1.902$

Check Your Progress-2

4. $x_1 = 9.2308$, $x_2 = -24.153$, $x_3 = 6.0769$
5. $x_1 = 175.8462$, $x_2 = 17.4615$, $x_3 = -38.8462$

3.6 GLOSSARY

5. Converge means to meet at a point.
6. Nontrivial means to have value of atleast one variable not equal to zero.
7. Augmented Matrix is a matrix formed by placing to two matrix side by side.

3.7 ASSIGNMENT

1. Compare and Contrast Between
   a) Jacobi's method ad Gauss-Seidel method
   b) Gauss elimination method and Gauss Jordan method
2. Solve the following system of linear equations

$$3x_1 + x_2 + x_3 = 9$$
$$2x_1 + 3x_2 - 3x_3 = -9$$
$$8x_1 + 4x_2 - 6x_3 = 5$$

   Using Matrix inversion method.

3.8 ACTIVITY

1. Solve the following system of linear equations

   $3x_1 + x_2 + x_3 = 7$

   $2x_1 + 3x_2 - 3x_3 = -11$

   $8x_1 + 4x_2 - 6x_3 = 9$

   Using Any 3 methods of eliminations.

3.9 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012.

UNIT 4: INTERPOLATION

Unit Structure

TRODUCTION

Suppose we have

$$y = f(x) \qquad x_1 \leq x \leq x_n$$

As a result, at least one value of y exists for each value of x in the range $x_1 \leq x \leq x_n$. If f(x) is known explicitly, one can directly compute f(x) for every value of x in the range $x_1 \leq x \leq x_n$.

Suppose f(x) is not explicitly known, and only a set of values (x_1, y_1), (x_2, y_2), (x_3, y_3),..., (x_n, y_n) satisfy the relation y = f(x) are known. This requires finding a simple function, say g(x) such that f(x) and g(x) agree at the specified values. This process is known as Interpolation.

There are numerous ways to interpolate the function g(x); parabolic or polynomial interpolation is used when g(x) is a function that is a polynomial; and trigonometric interpolation when g(x) is a trigonometric series. It is always practical to choose g(x) as the simplest function that represents the given function using the given interval, as g(x) may be any series of exponential functions, Legendre polynomials, Bessel functions, etc. Almost all standard formulas for interpolation involve polynomials, since polynomials are the simplest functions.

The concept of interpolation is essential in numeric analysis, as it provides the theoretical framework for calculating formulae for numerical differentiation and numerical integration as well as solving differential equations. As part of this chapter, we will study the Lagrangian interpolations, finite differences, difference tables, and Newton's methods of interpolation, along with an examination of error propagation in difference tables.

4.1 Lagrange Interpolation

To obtain a general formula of Lagrangian interpolation, let us consider a second order polynomial of type

$$y(x) = a_1(x - x_2)(x - x_3) + a_2(x - x_1)(x - x_3) + a_3(x - x_1)(x - x_2)$$

which passes through points (x_1, y_1), (x_2, y_2), (x_3, y_3), where a_1, a_2, and a_3 are unknown constants whose values can be determined by

At $x = x_1$

$$y(x_1) = a_1(x_1 - x_2)(x_1 - x_3)$$

$$\Rightarrow a_1 = \frac{y_1}{(x_1 - x_2)(x_1 - x_3)}$$

At $x = x_2$

$$y(x_2) = a_2(x_2 - x_1)(x_2 - x_3)$$

$$\Rightarrow a_2 = \frac{y_2}{(x_2 - x_1)(x_2 - x_3)}$$

At $x = x_3$

$$y(x_3) = a_3(x_3 - x_1)(x_3 - x_1)$$

$$\Rightarrow a_3 = \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}$$

On Substituting the values of a_1, a_2 , and a_3 in the equation above we get;

$$\Rightarrow y(x) = y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + y_3 \frac{(x-x_1)(x-x_3)}{(x_3-x_2)(x_3-x_1)}$$

And after using the Product and summation notations, The above equation can be also written as

$$\Rightarrow y(x) = \sum_{i=1}^{3} y_i \prod_{\substack{j=1 \\ j\neq i}}^{3} \frac{(x-x_j)}{(x_i-x_j)}$$

The above equation is a second order polynomial equation passing through points (x_1, y_1), (x_2, y_2), and (x_3, y_3).

So, General Formula for Lagrangian Polynomial is given by

$$\Rightarrow y(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}.$$

Example 4.1

Find Solution using Lagrange's Interpolation formula

x	2	2.5	3
y	0.6932	0.9163	1.0986

x = 2.7

Finding option 1. Value f (2)

Solution:

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

$$y(2.7) = \frac{(2.7-2.5)(2.7-3)}{(2-2.5)(2-3)} \times 0.6932 + \frac{(2.7-2)(2.7-3)}{(2.5-2)(2.5-3)} \times 0.9163 + \frac{(2.7-2)(2.7-2.5)}{(3-2)(3-2.5)} \times 1.0986$$

$$y(2.7) = \frac{(0.2)(-0.3)}{(-0.5)(-1)} \times 0.6932 + \frac{(0.7)(-0.3)}{(0.5)(-0.5)} \times 0.9163 + \frac{(0.7)(0.2)}{(1)(0.5)} \times 1.0986$$

$$y(2.7) = (-0.12) \times 0.6932 + 0.84 \times 0.9163 + 0.28 \times 1.0986$$

$y(2.7)=0.9941$

Solution of the polynomial at point 2.7 is $y(2.7)=0.9941$

Example 4.2

Using the following table find f(x) as polynomial in x

x	-1	0	3	6	7
f(x)	3	-6	39	822	1611

Solution:

Lagrange's Interpolating Polynomial
The value of x at you want to find $Pn(x):x=1$

$$y(x)=\frac{(x-x1)(x-x2)(x-x3)(x-x4)}{(x0-x1)(x0-x2)(x0-x3)(x0-x4)}\times y0+\frac{(x-x0)(x-x2)(x-x3)(x-x4)}{(x1-x0)(x1-x2)(x1-x3)(x1-x4)}\times y1+\frac{(x-x0)(x-x1)(x-x3)(x-x4)}{(x2-x0)(x2-x1)(x2-x3)(x2-x4)}\times y2+\frac{(x-x0)(x-x1)(x-x2)(x-x4)}{(x3-x0)(x3-x1)(x3-x2)(x3-x4)}\times y3+\frac{(x-x0)(x-x1)(x-x2)(x-x3)}{(x4-x0)(x4-x1)(x4-x2)(x4-x3)}\times y4$$

$$y(1)=\frac{(1-0)(1-3)(1-6)(1-7)}{(-1-0)(-1-3)(-1-6)(-1-7)}\times 3+\frac{(1--1)(1-3)(1-6)(1-7)}{(0--1)(0-3)(0-6)(0-7)}\times -6+\frac{(1--1)(1-0)(1-6)(1-7)}{(3--1)(3-0)(3-6)(3-7)}\times 39$$
$$+\frac{(1--1)(1-0)(1-3)(1-7)}{(6--1)(6-0)(6-3)(6-7)}\times 822+\frac{(1--1)(1-0)(1-3)(1-6)}{(7--1)(7-0)(7-3)(7-6)}\times 1611$$

$$y(1)=\frac{(1)(-2)(-5)(-6)}{(-1)(-4)(-7)(-8)}\times 3+\frac{(2)(-2)(-5)(-6)}{(1)(-3)(-6)(-7)}\times -6+\frac{(2)(1)(-5)(-6)}{(4)(3)(-3)(-4)}\times 39+\frac{(2)(1)(-2)(-6)}{(7)(6)(3)(-1)}\times 822$$
$$+\frac{(2)(1)(-2)(-5)}{(8)(7)(4)(1)}\times 1611$$

$$y(1)=(-0.2679)\times 3+0.9524\times -6+0.4167\times 39+(-0.1905)\times 822+0.0893\times 1611$$

$$y(1)=-3$$

Check Your Progress-1

Question 1: Using the data sin(0.1) = 0.09983 and sin(0.2) = 0.19867, find an approximate value of sin(0.15) by Lagrange interpolation.

Question 2: Give two uses of interpolating polynomials.

Question 3: Write the property satisfied by Lagrange fundamental polynomials in x.

4.2 Finite Differences and Difference Table

As the name implies, finite differences are comparisons between functions or comparisons between the results of previous comparisons.

There are four types of differences:

1. Forward Difference
2. Backward Difference
3. Divided Difference

1. Forward Differences: If $y_1, y_2, ..., y_n$ denotes a set of values of any function $y = f(x)$, then the differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, ..., \Delta y_{n-1}$ are termed as the first order forward differences. Here, Δ is the first order forward difference operator. The general expressions for first forward differences are

$$\Delta y_i = y_{i+1} - y_i$$

Similarly, 2nd and higher order differences are as

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i$$

and $$\Delta^p y_i = \Delta^{p-1} y_{i+1} + \Delta^{p-1} y_i$$

Further, these differences can be expressed in terms of the entries as:

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$

and so, on

$$\Delta^n y_0 = {}^nC_0 y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} + \cdots + (-1)^n y_0$$

The following difference table shows the differences of all orders:

TABLE 13.1: DIAGONAL FORWARD DIFFERENCE TABLE

Value of x	*Value of y*	*Ist diff*		*2nd diff*	*3rd diff*		*4th diff*	*5th diff*
x_0	y_0							
		Δy_0						
x_1	y_1		=>	$\Delta^2 y_0$				
		Δy_1			$\Delta^3 y_0$			
x_2	y_2		=>	$\Delta^2 y_1$		=>	$\Delta^4 y_0$	$\Delta^5 y_0$
		Δy_2			$\Delta^3 y_1$			
x_3	y_3		=>	$\Delta^2 y_2$		=>	$\Delta^4 y_1$	
		Δy_3			$\Delta^3 y_2$			
x_4	y_4		=>	$\Delta^2 y_3$				
x_5	y_5	Δy_4						

Backward Differences:

The differences $y_1 — y_0$, $y_2 — y_1, \ldots, y_n — y_{n-1}$ when denoted by , $Vy_1, Vy_2, \ldots, Vy_n$ are called the first order backward differences. Here V (Nebla) stands for backward difference operator. The general expression for first order backward differences is $Vy_i = y_i - y_{i-1}$ and thus the subsequent higher order backward differences are $V^2y_i = Vy_i — Vy_{i-1}$, $V^3y_i = V^2y_i - V^2y_{i-1}$ and so on. The following difference table shows how the differences of all orders are formed:

TABLE 13.2. DIAGONAL BACKWARD DIFFERENCE TABLE.

Value of x	*Value of y*	*1st diff*	*2nd diff*	*3rd diff*	*4th diff*	*5th diff*
X0	*Y0*					
		V'yl				
X1	Y1		$=>V^2v_2$			
		V'yz		V^3y_3		
X2	Y2		$=> V^2y_3$		$=> V^4y_4$	
		V'y3		V^3y_4		V^5y_5
X3	Y3		$=> V^2y_4$		$=> V^4y_5$	
		V'y4		V^3y_5		
X4	Y4		$=> V^2y_5$			
		V'y5				
X5	Y5					

3. Divided Differences :

The differences $\frac{y_2 - y_1}{x_2 - x_1}, \frac{y_3 - y_2}{x_3 - x_2}, \ldots, \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$ when denoted by $\Delta_d y_1, \Delta_d y_2, \Delta_d y_3, \ldots, \Delta_d y_{n-1}$ are called first order divided differences. Here Δ_d is called te divided difference operator. And the differences of the first order divided differences and are denoted as $\Delta^2_d y_1, \Delta^2_d y_2, \Delta^2_d y_3, \ldots, \Delta^2_d y_n$.

z	$f(z)$	First divided differences	Second divided differences
$z_0 = x_0$	$f[z_0] = f(x_0)$		
		$f[z_0, z_1] = f'(x_0)$	
$z_1 = x_0$	$f[z_1] = f(x_0)$		$f[z_0, z_1, z_2] = \frac{f[z_1, z_2] - f[z_0,}{z_2 - z_0}$
		$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1}$	
$z_2 = x_1$	$f[z_2] = f(x_1)$		$f[z_1, z_2, z_3] = \frac{f[z_2, z_3] - f[z_1,}{z_3 - z_1}$
		$f[z_2, z_3] = f'(x_1)$	
$z_3 = x_1$	$f[z_3] = f(x_1)$		$f[z_2, z_3, z_4] = \frac{f[z_3, z_4] - f[z_2,}{z_4 - z_2}$
		$f[z_3, z_4] = \frac{f[z_4] - f[z_3]}{z_4 - z_3}$	
$z_4 = x_2$	$f[z_4] = f(x_2)$		$f[z_3, z_4, z_5] = \frac{f[z_4, z_5] - f[z_3,}{z_5 - z_3}$
		$f[z_4, z_5] = f'(x_2)$	
$z_5 = x_2$	$f[z_5] = f(x_2)$		

4.3 NEWTON'S METHOD OF INTERPOLATION

4.3.1 Newton's Forward Difference Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, \ldots, y_n$ corresponding to the values $x_0, x_1, \ldots, x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih$ $(i = 0, 1, \ldots)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \ldots, y(x_n) = y_n$. We can write

$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \ldots + a_n(x - x_0)(x - x_1)\ldots$
$(x - x_{n-1})$

Putting $x = x_0, x_1, x_2, \ldots, x_n$ successively in above equation we get
$y_0 = a_0$, $y_1 = a_0 + a_1(x_1 - x_0)$, $y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$ and so on.

From these, we find that $a_0 = y_0$, $\Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore a_1 = \frac{1}{h}\Delta y_0$$

Also
$$\Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$= a_1 h + a_2 hh = \Delta y_0 + 2h^2 a_2$$
$$\therefore a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!h^2}\Delta^2 y_0$$

$$similiarly; a_3 = \frac{1}{3!h^3}\Delta^3 y_0$$

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots\ldots \quad (2)$$

Now if it is required to evaluate y for $x = x_0 + ph$, then

$(x - x_0) = ph$, $x - x_1 = x - x_0 - (x - x_0) = ph - h = (p-1)h$,

$(x - x_0) = x - x_0 - (x - x_0) = (p-1)h - h = (p-2)h$ etc.

Hence, writting $y(x) = y(x_0 + ph) = y_p$, (2) becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-(n-1))}{n!}\Delta^n y_0 \quad (3)$$

It is called Newton's forward interpolation formula as (3) contains y_0 and the forward differences of y_0.

For any real number p, we have defined E such that

$$E^p f(x) = f(x+ph)$$

$$y_p = f(x_0+ph) = E^p f(x_0) = (1+\Delta)^p y_0 \qquad [E = 1+\Delta]$$

$$= \{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\} y_0 \qquad (4)$$

[Using binomial theorem]

i.e. $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

If $y = f(x)$ is a polynomial of the nth degree, then $\Delta^{n+1} y_0$ and higher dif ferences will be zero.

Hence (4) will become

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)\ldots(p-(n-1))}{n!}\Delta^n y_0$$

Which is same as (3)

4.3.2 Newton's Backward Interpolation Formula

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0+h, x_0+2h, \cdots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_n + ph$, where p is any real number. Then we have

$$y_p = f(x_n+ph) = Ep\, f(x_n) = (1-\nabla)^{-p} y_n \qquad [E^{-1} = 1-\nabla]$$

$$= [1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \ldots] y_n$$

[using binomial theorem]

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots$$

It is called *Newton's backward interpolation formula* as (1) contains y_n and backward differences of y_n.

Example 4.3

Find Solution using Newton's Backward Difference formula

x	f(x)
0	1
1	0
2	1
3	10

x = 4

Finding option 1. Value f(2)

Solution:

The value of table for *x* and *y*

x	0	1	2	3
y	1	0	1	10

Newton's backward difference interpolation method to find solution

Newton's backward difference table is

x	**y**	**∇y**	**∇2y**	**∇3y**
0	1			
		-1		
1	0		2	
		1		**6**
2	1		**8**	
		9		

3	**10**			

The value of x at you want to find the $f(x)$:x=4

$h = x1\text{-}x0 = 1\text{-}0 = 1$

$p = x\text{-}xn\ h = 4 - 31 = 1$

Newton's backward difference interpolation formula is
$y(x) = yn + p\nabla yn + p(p+1)2!\cdot\nabla 2yn + p(p+1)(p+2)3!\cdot\nabla 3yn$

$y(4) = 10 + 1\times9 + 1(1+1)2\times8 + 1(1+1)(1+2)6\times6$

$y(4) = 10 + 9 + 8 + 6$

$y(4)=33$

Solution of newton's backward interpolation method $y(4)=33$

Example 4.4

In the table below the values of y are consecutive terms of a series of which the number 21.6 is the 6^{th} term. Find the 1^{st} and 10^{th} terms of the series.

X	3	4	5	6	7	8	9
Y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Solution:
Newton's backward difference table is

x	y	∇y	$\nabla 2y$	$\nabla 3y$	$\nabla 4y$
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3		0
		9.1		0.6	

6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		**0**
		16.9		**0.6**	
8	51.2		**4.8**		
		21.7			
9	**72.9**				

The value of x at you want to find the $f(x)$:$x=1$

$h=x_1-x_0=4-3=1$

4.3.3 Newton's Divided Difference Formula

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of y=f(x) corresponding to the arguments $x_0, x_1, x_2, \ldots, x_n$. Then from the definition be the values of of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

So that $y = y_0 + (x - x_0)[x, x_0]$

Again $$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1} \qquad (2)$$

which gives $[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]$$

Also $$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2]$$
$$+ (x - x_0)(x - x_1) \ldots (x - x_n)[x, x_0, x_1, \ldots x_n]$$
$$+ (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2] + \ldots \qquad (3)$$

which is called *Newton's general interpolation formula with divided differences*.

Example 4.5

Find Solution using Newton's Divided Difference Interpolation formula

x	f(x)
300	2.4771
304	2.4829
305	2.4843
307	2.4871

x = 301
Finding option 1. Value f(2)

Solution:
The value of table for *x* and *y*

x	300	304	305	307
y	2.4771	2.4829	2.4843	2.4871

Numerical divided differences method to find solution

Newton's divided difference table is

x	**y**	**1*st* order**	**2*nd* order**
300	2.4771		
		0.0014	
304	2.4829		0
		0.0014	
305	2.4843		0
		0.0014	
307	2.4871		

The value of *x* at you want to find the $f(x)$:x=301

Newton's divided difference interpolation formula is

$f(x) = y_0 + (x-x_0)\, f[x_0,x_1] + (x-x_0)(x-x_1)\, f[x_0, x_1, x_2]$

$y(301) = 2.4771 + (301-300)\times 0.0014 + (301-300)(301-304)\times 0$

$y(301) = 2.4771 + (1)\times 0.0014 + (1)(-3)\times 0$

$y(301) = 2.4771 + 0.0014 + 0$

$y(301)=2.4785$

Solution of divided difference interpolation method $y(301)=2.4785$

Example 4.6

Find Solution using Newton's Divided Difference Interpolation formula

x	f(x)
2	0.69315
2.5	0.91629
3	1.09861

x = 2.7
Finding option 1. Value f(2)

Solution:
The value of table for *x* and *y*

x	2	2.5	3
y	0.6932	0.9163	1.0986

Numerical divided differences method to find solution

Newton's divided difference table is

x	**y**	1*st* order	2*nd* order
2	0.6932		

		0.4463	
2.5	0.9163		-0.0816
		0.3646	
3	1.0986		

The value of x at you want to find the $f(x)$:x=2.7

Newton's divided difference interpolation formula is
$f(x) = y_0 + (x-x_0)\, f\,[x_0,x_1] + (x-x_0)(x-x_0)\, f\,[x_0, x_1, x_2]$

$y(2.7) = 0.6932 + (2.7-2) \times 0.4463 + (2.7-2)(2.7-2.5) \times -0.0816$

$y(2.7) = 0.6932 + (0.7) \times 0.4463 + (0.7)(0.2) \times -0.0816$

$y(2.7) = 0.6932 + 0.3124 - 0.0114$

$y(2.7) = 0.9941$

Solution of divided difference interpolation method $y(2.7)$=0.9941

Check Your Progress-2

1. In an experiment the pressure of gas is recorded as 50, 53, 57, 68, and 82 at Intervals of two second . The first recording was done at zero second. Obtain an estimate of the pressure of the gas at the time of 2.5 seconds and 6.2 seconds.

2. The distance covered car in a given time (in minutes) is given in the following table:

X	: 1	2	3
F(x)	: 1	20	40

4.7 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 5: SPLINE INTERPOLATION

Unit Structure

5.0 INTRODUCTION:

It is usually difficult to analyze numerical data in the real world. It would be difficult to obtain and highly unwieldy to construct any function that effectively correlates the data. As a result, the cubic spline was developed. Using this process, a series of cubic polynomials are fitted between each data point, ensuring that the curve obtained is continuous and smooth. In this way, rates and cumulative changes over time can be calculated using cubic splines. Although a more robust form could accommodate unequally spaced points as well, we will only discuss splines that interpolate equally spaced data points in this brief introduction.

5.1 METHOD TO FIND THE SPLINE INTERPOLATION:

Fitting a piecewise function of the form is the essential idea

:
:
:

Where a_i is a third degree polynomial defined by

$$a_i(x) = p_i(x - x_i)^3 + q_i(x - x_i)_2 + r_i(x - x_i) + s_i$$

For i = 1,2,3, …, n-1

The first and second derivatives of these n-1 equations are fundamental to this process and they are

$$a_i^{'}(x) = 3p_i(x - x_i)^2 + 2q_i(x - x_i) + r_i$$
$$a_i^{''}(x) = 6p_i(x - x_i) + 2q_i$$

Example 5.1

Calculate Cubic Splines

X	0	1	2
Y	10	20	30

Then find: y(0.5), y'(4)

Solution:

x	0	1	2
y	10	20	30

Cubic spline formula is

$$f(x)=\frac{(x_i-x)^3}{6h}M_{i-1}+\frac{(x-x_{i-1})^3}{6h}M_i+\frac{(x_i-x)}{h}(y_{i-1}-\frac{h^2}{6}M_{i-1})+\frac{(x-x_i)}{h}(y_i-\frac{h^2}{6}M_i)\rightarrow(1)$$

We have, $M_{i-1}+4M_i+M_{i+1}=\frac{6}{h^2}(y_{i-1}-2y_i+y_{i+1})\rightarrow(2)$

Here h=1,n=2

M_0 =0,M_2 = 0

Substitute i=1 in equation (2)

$$M_0+4M_1+M_2=\frac{6}{h^2}(y_0+2y_1+y_2)$$

$$\Rightarrow 0+4M_1+0=\frac{6}{1}(10-40+30)$$

$\Rightarrow 4M_1 = 0$

$\Rightarrow M_1 = 0$

Substitute i=1 in equation (1), we get cubic spline in 1*st* interval $[x_0,x_1]$=[0,1]

$$f_1(x)=\frac{(x_1-x)^3}{6h}M_0+\frac{(x-x_0)^3}{6h}M_1+\frac{(x_1-x)}{h})(y_0-\frac{h^2}{6}M_0)+\frac{(x-x_0)}{h}(y_1-\frac{h^2}{6}M_1)$$

$$f_1(x)=\frac{(1-x)^3}{6}.0+\frac{(x-0)^3}{6}M_1+\frac{(1-x)}{1})(10-\frac{1}{6}.0)+\frac{(x-0)}{1}(20-\frac{1}{6}.0)$$

$f_1(x)=10x+10$, for $0\le x\le 1$

Substitute i=2 in equation (1), we get cubic spline in 2*nd* interval $[x_1,x_2]$=[1,2]

$$f_2(x)=\frac{(x_2-x)^3}{6h}M_1+\frac{(x-x_1)^3}{6h}M_2+\frac{(x_2-x)}{h})(y_1-\frac{h^2}{6}M_1)+\frac{(x-x_1)}{h}(y_2-\frac{h^2}{6}M_2)$$

$$f_2(x)=\frac{(2-x)^3}{6}.0+\frac{(x-1)^3}{6}+\frac{(2-x)}{1})(20-\frac{1}{6}.0)+\frac{(x-1)}{1}(30-\frac{1}{6}.0)$$

$f_2(x)=10x+10$, for $1\le x\le 2$

For $y(0.5)$, 0.5∈[0,1], so substitute x=0.5 in $f_1(x)$, we get

$f_1(0.5)$=15

Check Your Progress – 1

1. Calculate Cubic Splines

X	0	-1	-2	-3
Y	1	2	3	4

Then find: y(-1.5), y'(2)

2. Calculate Cubic Splines

X	0	10	20	30
Y	20	40	60	80

Then find: y(50), y'(30)

5.2 TYPES OF SPLINES:

5.2.1. Natural Spline:

At the endpoints of this first spline type, the second derivative is zero.

$$M_1 = M_n = 0$$

As a result, the spline extends outside the endpoints. It is possible to adapt the matrix for determining M_1-M_n values accordingly.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & & & . & . & . & . \\ . & . & . & . & & & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ M_2 \\ M_3 \\ M_4 \\ . \\ . \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \\ 0 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_3 + y_5 \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

The first and last columns of this matrix can be eliminated since they correspond to the M_1 and M_n values, both of which are zero.

$$\begin{bmatrix} 4 & 1 & 0 & . & . & 0 & 0 & 0 \\ 1 & 4 & 1 & . & . & 0 & 0 & 0 \\ 0 & 1 & 4 & . & . & 0 & 0 & 0 \\ . & . & . & & & . & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & 4 & 1 & 0 \\ 0 & 0 & 0 & . & . & 1 & 4 & 1 \\ 0 & 0 & 0 & . & . & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \mathrm{M}_2 \\ \mathrm{M}_3 \\ M_4 \\ . \\ . \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

This produces an n-2 by n-2 matrix that determines the remaining solutions for M_2 through M_{n-1}. Now the spline is unique.

5.2.2 Parabolic Runout Spline:

For a parabolic spline, the second derivative at each endpoint, M_1 and M_n, must equal M_2 and M_{n-1}.

$$M_1 = M_2$$

$$M_n = M_{n-1}$$

As a result, the curve becomes parabolic at the endpoint. Periodic and exponential data can benefit from cubic splines of this type.

The Matrix Equation for this type of Spline is :

$$\begin{bmatrix} 5 & 1 & 0 & . & . & 0 & 0 & 0 \\ 1 & 4 & 1 & . & . & 0 & 0 & 0 \\ 0 & 1 & 4 & . & . & 0 & 0 & 0 \\ . & . & . & & & . & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & 4 & 1 & 0 \\ 0 & 0 & 0 & . & . & 1 & 4 & 1 \\ 0 & 0 & 0 & . & . & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ . \\ . \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

M_1 and M_n already being determined now we can find the value of M_2 through M_{n-1}.

5.2.3 Cubic Runout Spline:

Last but not least, this type of spline has te most extreme endpoint behaviour. Assumin M_1 to be $2M_2 - M_3$, and M_n to be $2M_{n-1} - M_{n-2}$, it degrades to a single cubic over the last two intervals.

The Matrix equation for such equation is

$$\begin{bmatrix} 6 & 0 & 0 & . & . & 0 & 0 & 0 \\ 1 & 4 & 1 & . & . & 0 & 0 & 0 \\ 0 & 1 & 4 & . & . & 0 & 0 & 0 \\ . & . & . & & & . & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & 4 & 1 & 0 \\ 0 & 0 & 0 & . & . & 1 & 4 & 1 \\ 0 & 0 & 0 & . & . & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ . \\ . \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ . \\ . \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

You should also be aware that other interpolating curves such as periodic spline and clamped spline also exist. The three splines described in this work are simply those we chose to examine. They are neither necessarily better than, nor more widely used than , the other types of spline.

5.3 LET US SUM UP:

In this chapter we discuss what spline interpolation is, and understood three types of splines:

- Natural Spline
- Parabolic Runout Spline
- Cubic Runout Spline

5.4 Suggested Answer for Check Your Progress

Check Your Progress-1

1. $f'(2) = 3$

2. $f'(30) = 2$

5.5 GLOSSARY:

1. A spline is a special function defined piecewise by polynomials.

5.6 ASSIGNMENT:

1. Calculate Cubic Splines

X	1	2	3	4
Y	1	5	11	8

Then find: y(1.5), y'(2)

2. Calculate Cubic Splines

X	1	2	3	4
Y	0	8	20	37

Then find: y(0.8), y'(2.7)

5.7 ACTIVITY:

1. Calculate Cubic Splines

X	0	1	2
Y	-5	-4	3

Then find: y(0.5)

2. Calculate Cubic Splines

X	1	2	3	4	5
Y	0	1	0	1	0

Then Find: y(1.5), y'(4)

4.7 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

Computer Oriented Numerical Method

BLOCK 3: Curve Fitting

Block Structure

BLOCK1:

UNIT 1: METHOD OF LEAST SQUARE

Unit Structure

1.0 INTRODUCTION:

Most often, in practice, two or more variables are found to be related. The relationship between these variables can be expressed mathematically by determining an equation connecting them. In regression analysis or curve fitting, the problem is to find equations for approximating curves that match a set of data. Curves can be approximated using a variety of methods, but the least squares method is the most popular and useful.

In general, curve fitting relies on determining the continuous function

$$y = f(x)$$

Which results in the best fit for the given set of values of (x, y), (x1, y1), (x2, y2), ... (xn, yn). A measurer may be able to guess in advance what form f(x) will take based on their physical laws or theories. For large numbers of f(x), the useful practice is to plot all or some of the points, and the appearance of the plotted points can indicate what the particular form of f(x) is.

This chapter discusses how to fit curves using points as input which are given in form of a table.

1.1 METHOD OF LEAST SQUARES:

For any data set described by an equation, the least square method finds the best-fit line. It involves calculating the trend numerically by reducing the sum of squares of the residual parts of the curve or line. In the least square method, the fitting equations to derive the curve are derived by regression analysis.

If x_1, x_2, x_3, …, x_n be the values of the independent values for variable x and similarly y_1, y_2, y_3, …, y_n be the dependent values for variable y. then the graph obtained through these points is shown below:

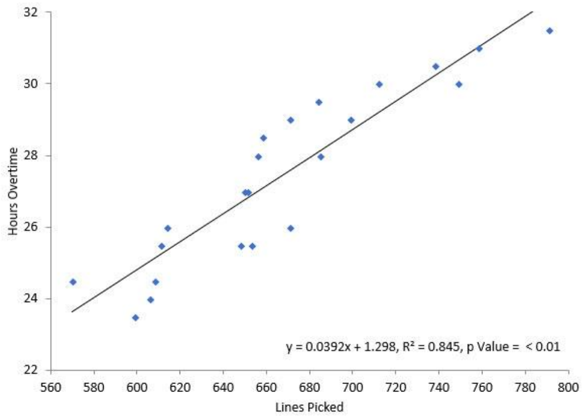

Fig 1.1 Scatter Diagram

Let's assume that

$$y = f(x)$$

is the approximation to the function.
Then the error between the value of dependent variable y from the approximations made above and the actual values of y_s are,

$$d_1 = y_1 - f(x_1)$$
$$d_2 = y_2 - f(x_2)$$
$$d_3 = y_3 - f(x_3)$$
$$\vdots$$
$$d_n = y_n - f(x_n)$$

And if we find the sum of square of all the derivatives, then this sum would either be maximum or minimum, if the partial derivative $\frac{\partial S}{\partial a} = 0$, in which one of the unknowns used in approximation y=f(x).

And by this method to equate all the derivatives by zero we obtain a system of non-homogeneous linear equations which can be solved using methods learned in previous chapters.

Hence we can say that the least square method states that

A curve is said to be the best curve if the sum of derivations of the individual points of the curve is minimum.

1.2 FITTING A STRAIGHT LINE:

Let $x_1, x_2, x_3, \dots, x_n$ be the values of the independent variable x and $y_1, y_2, y_3, \dots, y_n$ be the values of the dependent variable y.

And let $y = f(x) = a_1 x + a_0$ be the equation of a straight line which is the simplest type of curve for approximation. In the above equation $a_{1\backslash 0}$ is the intercept of the line, and a_1 is the slope and this points are also known as regression coefficients or slope coefficients. And the line is known as regression line of the y on x. And later we can find the equation of line by calculating the regression coefficient.

To find the equation of line, first we fins the sum of derivations;

$$S = \sum_{i=1}^{n} d_i^2$$

$$= \sum_{i=1}^{n} (y - f(x))^2$$

$$= \sum_{i=1}^{n} (y - a_2 x - a_1)^2$$

And for S to be minimum;

$$\frac{\partial S}{\partial a_0} = \sum_{i=1}^{n} 2(y_i - a_1 x_i - a_0)(-1) = 0$$

And

$$\frac{\partial S}{\partial a_1} = \sum_{i=1}^{n} 2(y_i - a_1 x_i - a_0)(-x_i) = 0$$

And on simplifying ahead we obtain ;

$$na_0 + a_1(\sum x_i) = \sum y_i$$

$$a_0(\sum x_i) + a_1(\sum x_i^2) = \sum x_i y_i$$

Here we used $\sum$ instead of $\sum_{i=1}^{n}$ for simplicity and easy understanding.

On solving above two equations using Cramer's rule we can conclude that the solutions are:-

$$a_0 = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$$

Example 1.1

Fit a straight line y = a + bx using the following data

x	5	4	3	2	1
y	1	2	3	4	5

Solution:

Straight line equation is *y*=*a*+*bx*.

The normal equations are

$\sum y = an + b \sum x$

$\sum xy = a\sum x + b\sum x^2$

The values are calculated using the following table

x	y	x^2	xy
5	1	25	5
4	2	16	8

3	3	9	9
2	4	4	8
1	5	1	5
---	---	---	---
$\sum x=15$	$\sum y=15$	$\sum x^2=55$	$\sum xy=35$

Substituting these values in the normal equations
$5a+15b=15$

$15a+55b=35$

Solving these two equations using Elimination method,

$5a+15b=15$

$\rightarrow 5(a+3b)=5\cdot 3$

$\rightarrow a+3b=3$

and $15a+55b=35$

$\rightarrow 5(3a+11b)=5\cdot 7$

$\rightarrow 3a+11b=7$

$a+3b=3 \rightarrow (1)$

$3a+11b=7 \rightarrow (2)$

Equation$(1)\times 3 \Rightarrow 3a+9b=9$

Equation$(2)\times 1 \Rightarrow 3a+11b=7$

Substracting $\Rightarrow -2b=2$

$\rightarrow 2b=-2$

$\rightarrow b=-22$

$\rightarrow b=-11$

$\rightarrow b=-1$

Putting $b=-1$ in equation (1), we have

$a+3(-1)=3$

$\rightarrow a=3+3$

$\rightarrow a=6$

$\therefore a=6 \text{ and } b=-1$

Now substituting this values in the equation is $y=a+bx$, we get

$y=6-x$

Example 1.2

Fit a straight line to the following data on production.

x	1996	1997	1998	1999	2000
y	40	50	62	58	60

Solution:

Straight line equation is $y=a+bx$.

The normal equations are

$\sum y=an+b\sum x$

$\sum xy=a\sum x+b\sum x^2$

The values are calculated using the following table

X	Y	$x=(X-1998)$	x^2	$x\cdot y$
1996	40	-2	4	-80
1997	50	-1	1	-50
1998	62	0	0	0
1999	58	1	1	58
2000	60	2	4	120
---	---	---	---	---
$\sum X=9990$	$\sum y=270$	$\sum x=0$	$\sum x^2=10$	$\sum x\cdot y=48$

Substituting these values in the normal equations

$5a+0b=270$

$0a+10b=48$

Solving these two equations using Elimination method,

$5a=270$

$\rightarrow a=54$

and $10b=48$

$\rightarrow 5b=24$

$\therefore 5a=270\rightarrow(1)$

$10b=48\rightarrow(2)$

Taking equation (1), we have

$\Rightarrow 5a=270$

$\Rightarrow a=2705$

$\Rightarrow a=54\rightarrow(3)$

Taking equation (2), we have

$\Rightarrow 10b=48$

$\Rightarrow b=4810$

$\Rightarrow b=245\rightarrow(4)$

$a=54 \text{and} b=245$

Now substituting this values in the equation is $y=a+bx$, we get

$y=54+4.8x$

$\therefore y=54+4.8(X\text{-}1998)$

Check Your Progress-1

1. Given a table of values for the function as

x:	1	2	3	4	5	6
y:	12	15	18	21	24	27

2. Given a table of the values for the function as

x:	0.1	0.2	0.3	0.4	0.5	0.6
y:	2	4	6	8	10	12

3. Given a table of the values for the function as

x:	3	6	9	12	15	18

y:	1	3	5	7	9	11

1.5 LET US SUM UP

In this chapter we learnt techniques of fitting a curve through a set of points given in form of tables.
Such as,

- Fitting a straight line

And in upcoming Chapters we will discuss more different techniques for curve fitting.

1.6 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

6. y = 9+3x
7. y=20x
8. y=-1+0.6667x

1.6 GLOSSARY

8. Regression is a measure of the relation between the mean value of one variable and corresponding values of other variables.
9. Algorithm is a procedure, a description of a set of the steps that can be used to solve a mathematical computation.

1.7 ASSIGNMENT

3. Find the least squares fit of the form $y = a_0 + a_1 x^2$ to the following data

x	-1	0	1	2
y	2	5	7	1

4. Find the curve of best fit of the type $y = ae^{bx}$ to the following data by the method of least squares:

x	1	5	7	9	12

y	15	25	12	11	7

1.8 ACTIVITY

1. Extend least squares method to fit a kth order polynomial. What is the matri of coefficients for the matrix of coefficients for the normal equations? Assuming an algorithm is available which will solve k simultaneous equations in k unknowns, write an algorithm for kth order polynomial fit for a given set of n points.

1.9 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.

- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 2 : CURVE FITTING

Unit Structure

a. **Introduction**

b. **Fitting a Polynomial**

c. **Fitting a Non-Linear Equation**

i. **Fitting a Geometric Curve**

ii. **Fitting an Exponential Curve**

d. **Let's Sum Up**

e. **Suggested Answer to Check Your Progress**

f. **Glossary**

g. **Assignment**

h. **Activity**

i. **Further Reading**

2.0 INTRODUCTION:

In this discussion about various fittings are explained.

2.1 FITTING A POLYNOMIAL:

Let the quadratic curve be represented by

$$y = a_0 + a_1 x + a_2 x^2$$

Which can also be represented by $\bar{y}_i$:

$$\bar{y}_i = a_0 + a_1 x_i + a_2 x_i^2$$

the sum of squares of the derivations is given by

$$S = \sum (y_i - \bar{y}_i)^2 = \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

Differentiating S with the respect to a_0, a_1, a_2 respectively and equating each one to zero we obtain the following normal equations:

$$na_0 + a_1 \sum x_i + a_2 \sum x_i^2 = \sum y_i$$

$$a_0 \sum x_i + a_1 \sum x_i^2 + a_2 \sum x_i^3 = \sum x_i y_i$$

$$a_0 \sum x_i^2 + a_1 \sum x_i^3 + a_2 \sum x_i^4 = \sum x_i^2 y_i$$

Three linear equations are given with three unknowns. In quadratic regression, these equations are called normal equations which can be solved by the Gauss elimination procedure discussed in the previous chapters.

The same idea can be applied to any polynomial with nth degree. The principle of Gauss elimination can be applied to solve an nth degree polynomial with (n + 1) simultaneous equations in (n + 1) unknowns. Unfortunately, these equations are often ill-conditioned in practice, and applying Gauss elimination straight away can be difficult. In these cases, numerical accuracy can be improved by using a method similar to the one used in linear regression.

Example 2.1

Fit second degree parabola equation y = a + bx + cx^2 using the following data

x	2	4	6	8	10	12
y	1	2	3	4	5	6

Solution:

The equation is $y=a+bx+cx^2$ and the normal equations are

$\sum y = an + b\sum x + c\sum x^2$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$

The values are calculated using the following table

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	4	4	8	16	8	16
3	6	9	27	81	18	54
4	8	16	64	256	32	128
5	10	25	125	625	50	250
6	12	36	216	1296	72	432
---	---	---	---	---	---	---
$\sum x=21$	$\sum y=42$	$\sum x^2=91$	$\sum x^3=441$	$\sum x^4=2275$	$\sum xy=182$	$\sum x^2y=882$

Substituting these values in the normal equations
$6a+21b+91c=42$

$21a+91b+441c=182$

$91a+441b+2275c=882$

Solving these 3 equations,
Total Equations are 3

$6a+21b+91c=42 \rightarrow (1)$

$21a+91b+441c=182 \rightarrow (2)$

$91a+441b+2275c=882 \rightarrow (3)$

Select the equations (1) and (2), and eliminate the variable a.

$6a+21b+91c=42 \quad \times 7 \rightarrow \quad 42a+147b+637c=294$

-

$21a+91b+441c=182 \times 2 \rightarrow \quad 42a+182b+882c=364$

$- 35b - 245c = -70 \rightarrow (4)$

Select the equations (1) and (3), and eliminate the variable a.

$6a+21b+91c=42 \quad \times 91 \rightarrow \quad 546a+1911b+8281c = 3822$

$91a+441b+2275c=882 \times 6 \rightarrow \quad 546a+2646b+13650c=5292$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad - 735b - 5369c = -1470 \rightarrow (5)$

Select the equations (4) and (5), and eliminate the variable *b*.

$-35b-245c=-70 \quad \times 21 \rightarrow \quad -735b-5145c=-1470$

$-735b-5369c=-1470 \times 1 \rightarrow \quad -735b-5369c=-1470$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 224c = 0 \quad \rightarrow (6)$

Now use back substitution method
From (6)
224c=0
$\Rightarrow c=0224=0$

From (4)
$-35b-245c=-70$
$\Rightarrow -35b-245(0)=-70$
$\Rightarrow -35b=-70$
$\Rightarrow b=-70-35=2$

From (1)
$6a+21b+91c=42$
$\Rightarrow 6a+21(2)+91(0)=42$
$\Rightarrow 6a+42=42$
$\Rightarrow 6a=42-42=0$
$\Rightarrow a=06=0$

Solution using Elimination method.
$a=0, b=2, c=0$

Now substituting this values in the equation is $y=a+bx+cx^2$, we get
$y=0+2x+0x^2$

Check Your Progress-1

1. Given a table for the function as

x:	1.0	1.5	2.0	2.5	3.0	5.0
y:	0.7	1.1	1.5	1.9	2.3	2.7

Find a second degree polynomial.

2. Given a table for the function as

x:	0.1	0.2	0.3	0.4	0.5	0.6

y:	0.7	1.4	2.1	2.8	3.5	4.2

3. Given a tale for the function as

x:	0.3	0.5	0.8	1.2	1.5	1.9
y:	1.8	2.3	2.8	3.4	4.0	4.8

2.2 FITTING A NON-LINEAR FUNCTION:

There are many times when experimental data follow other geometric shapes, such as exponential, trigonometric, parabola, ellipse, etc., that can be determined from the scatter diagram or other physical factors.

Most fitting methods require transforming these functions into linear form, but this is not always possible. In this section, we will examine some functions which can be transformed into linear form.

2.2.1 Fitting a Geometric Curve

Let the curve be described by the equation

$$y = ax^b + c$$

$$\Rightarrow y - c = ax^b$$

On applying logarithm on both side, we get

$$z = \log(y - c) = \log ax^b = \log a + b \log x$$

Or $$z = a_0 + a_1 t$$

Where $a_0 = \log a, a_1 = b$ and $t = \log x$

The above equations obtained, is in linear form. To obtain the normal equations for the above geometric curve, method used in straight line fitting are applied here too.

$$n \log a + (\sum \log x_i) b = \sum \log(y_i - c)$$

$$(\sum \log x_i) \log a + \sum (\log x_i)^2 b = \sum \log x_i \log(y_i - c)$$

After solving the above equation we get the solutions for a_0, a_1 .

$$a_0 = \frac{\sum \log y \sum (\log x)^2 - \sum \log x \sum \log x \log y}{n \sum (\log x)^2 - (\sum \log x)^2}$$

$$a_1 = \frac{n \sum \log x \log y - \sum \log x \sum \log y}{n \sum (\log x)^2 - (\sum \log x)^2}$$

Example 2.2

Calculate Fitting exponential equation ($y=ab^x$) - Curve fitting using Least square method

x	y
0.5	13.54
1.0	17.85
1.5	22.43
2.0	28.54
2.5	36.0

Solution:

The curve to be fitted is $y=ab^x$

taking logarithm on both sides, we get
$\log10(y)=\log10(a)+x\log10(b)$

$Y = A + Bx$
where $Y = \log10(y)$, $A = \log10(a)$,$B = \log10(b)$

which linear in y,x
So the corresponding normal equations are
$\sum Y = nA + B\sum x$

$\sum xY = A\sum x + B\sum x^2$

The values are calculated using the following table

x	Y	$Y=\log10(y)$	$x2$	$x\cdot Y$
0.5	13.54	1.1316	0.25	0.5658
1	17.85	1.2516	1	1.2516
1.5	22.43	1.3508	2.25	2.0262
2	28.54	1.4555	4	2.9109
2.5	36	1.5563	6.25	3.8908
---	---	---	---	---
$\sum x=7.5$	$\sum y=118.36$	$\sum Y=6.7458$	$\sum x^2=13.75$	$\sum x\cdot Y=10.6454$

Substituting these values in the normal equations
$5A+7.5B=6.7458$

$7.5A+13.75B=10.6454$

Solving these two equations using Elimination method,

$5a+7.5b=6.7458$
and $7.5a+13.75b=10.6454$
$\therefore 7.5a+13.75b=10.65$

$5a+7.5b=6.7458 \rightarrow (1)$
$7.5a+13.75b=10.6454 \rightarrow (2)$

equation$(1)\times 7.5 \Rightarrow 37.5a+56.25b=50.5935$

equation$(2)\times 5 \Rightarrow 37.5a+68.75b=53.227$

Substracting $\Rightarrow -12.5b=-2.6335$
$\Rightarrow 12.5b=2.6335$
$\Rightarrow b=2.633512.5$
$\Rightarrow b=0.21068$

Putting $b=0.21068$ in equation (1), we have
$5a+7.5(0.21068)=6.7458$
$\Rightarrow 5a=6.7458-1.5801$
$\Rightarrow 5a=5.1657$
$\Rightarrow a=5.16575$
$\Rightarrow a=1.03314$
$\therefore a=1.03314 \text{and} b=0.21068$

we obtain $A=1.0331, B=0.2107$

$\therefore a=anti\log 10(A)=anti\log 10(1.0331)=10.7929$
and $b=anti\log 10(B)=anti\log 10(0.2107)=1.6244$

Now substituting this values in the equation is $y=abx$, we get

$y=10.7929\cdot(1.6244)x$

2.2.2 Fitting an Exponential Curve

Let the exponential curve be described by the equation

$$y = ae^{bx} \text{ and } y = ae^{-bx}$$

Let's take the exponential curve of form

$$y = ae^{-bx}$$

On applying logarithm on both side, we get

$$z = \log y = \log ae^{-bx} = \log a + (-bx)$$

Here, let $a_0 = \log a$ and $a_1 = -b$

So, we can rewrite the equation as

$$z = a_0 + a_1 x$$

The above equations obtained, is in linear form. To obtain the normal equations for the above exponential curve, method used in straight line fitting are applied here too.

$$na_0 + (\sum x_i)a_1 = \sum z_i = \sum \log y_i$$

$$(\sum x_i)a_0 + (\sum x_i^2)a_1 = \sum x_i z_i = \sum x_i \log y_i$$

And, On solving above equations we get

$$a_0 = \frac{\sum \log y_i \sum x_i^2 - \sum x_i \sum x_i \log y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

$$a_1 = \frac{n\sum x_i \log y_i - \sum x_i \sum \log y_i}{n\sum x_i^2 - (\sum x_i)^2}$$

After we obtain the values of a_0 and a_1, we have to find the value of a and b using equations $a_0 = \log a$ and $a_1 = -b$ as

$$a_1 = -b$$

$$b = -a_1$$

Example 2.3

Fit exponential equation y = ae^{bx} using the following data.

X	0.0	0.5	1.0	1.5	2.0
Y	0.10	0.45	2.15	5.55	6.50

Solution:

The curve to be fitted is $y=ae^{bx}$

taking logarithm on both sides, we get

$\log_{10}(y) = \log_{10}(a) + bx\log_{10}(e)$

$Y = A + Bx$

where $Y = \log_{10}(y)$, $A = \log_{10}(a)$, $B = b\log_{10}(e)$

which linear in Y,x

So the corresponding normal equations are

$\sum Y = nA + B\sum x$

$\sum xY = A\sum x + B\sum x^2$

The values are calculated using the following table

x	y	$Y=\log_{10}(y)$	x^2	$x \cdot Y$
0	0.1	-1	0	0
0.5	0.45	-0.3468	0.25	-0.1734
1	2.15	0.3324	1	0.3324
1.5	5.55	0.7443	2.25	1.1164
2	6.5	0.8129	4	1.6258
---	---	---	---	---
$\sum x=5$	$\sum y=14.75$	$\sum Y=0.5429$	$\sum x^2=7.5$	$\sum x \cdot Y=2.9013$

Substituting these values in the normal equations
$5A+5B=0.5429$
$5A+7.5B=2.9013$

Solving these two equations using Elimination method,
$5a+5b=0.5429$
and $5a+7.5b=2.9013$
$\therefore 5a+7.5b=2.9$

$5a+5b=0.5429 \rightarrow (1)$
$5a+7.5b=2.9013 \rightarrow (2)$

Substracting $\Rightarrow -2.5b=-2.3584$
$\Rightarrow 2.5b=2.3584$
$\Rightarrow b=2.35842.5$
$\Rightarrow b=0.94336$

Putting $b=0.94336$ in equation (1), we have

$5a+5(0.94336)=0.5429$
$\Rightarrow 5a=0.5429-4.7168$
$\Rightarrow 5a=-4.1739$
$\Rightarrow a=-4.17395$
$\Rightarrow a=-0.83478$

$\therefore a = -0.83478$ and $b = 0.94336$

we obtain $A = -0.8348$, $B = 0.9434$

$\therefore a = \text{antilog}_{10}(A) = \text{antilog}_{10}(-0.8348) = 0.1463$

and $b = \dfrac{B}{\log_{10}(e)} = \dfrac{0.9434}{0.4343} = 2.1722$

Now substituting these values in the equation is $y=ae^{bx}$, we get

$y=0.1463 \cdot e^{2.1722x}$

Check Your Progress-2

1. Given a table of values for the function as

y:	2	5	9	11
x:	100	300	550	700

It's known that equation of type $y = ae^{bx}$ exists.
Find the best possible values of a and b.

2. Given a table of values for the function as

y:	2	4	6	8	10

x: 22.4 36.8 46.4 56.7 65.9

It's known that equation of type $y = ae^{bx}$ exists.

Find the best possible values of a and b.

2.3 LET US SUM UP

In this chapter we learnt techniques of fitting a curve through a set of points given in form of tables.

Such as,

- Fitting a Polynomial
- Fitting a Geometric Curve
- Fitting a Exponential Curve

2.4 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

6. $y = -0.1 + 0.8x + 0x^2$

7. y=7x

8. y=1.3632+1.6817x+0.0597x2

Check Your Progress-2

1. $y = 80.0433 \cdot e^{0.2089x}$

2. $y = 19.6832 \cdot e^{0.1295x}$

2.5 GLOSSARY:

1. **Intercept** is the point where a line crosses either x-axis or y-axis.
2. **Matrix** is a set of numbers arranged in rows and columns so as to form a rectangular array.

2.6 ASSIGNMENT:

1. Calculate Fitting second degree parabola - Curve fitting using Least square method

X	Y
1	2

3	4
5	6
7	8
9	10

2. Calculate Fitting exponential equation (y=ae^bx) - Curve fitting using Least square method

X	Y
11	30
15	35
19	40
23	45
27	50

Also Estimate y for x = 29

2.7 ACTIVITY:

1. Calculate Fitting exponential equation (y=ax^b) - Curve fitting using Least square method

X	Y
3	1
10	2
17	3
24	4
31	5

Also Estimate y for x = 8

2.8 FURTHER READING:

- M.K. Jain, S.R.K. Iyengar And R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.

- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012.

Computer Oriented Numerical Method

BLOCK4: Numerical Differentiation and Integration

Block Structure

BLOCK1:

UNIT1 Numerical Differentiation

Objectives, Differentiating a Function, Differentiating a Tabulated Function, Let Us Sum Up

UNIT2 Numerical Integration

Objectives, Newton-Cotes Integration Formulae, Trapezoidal Rule, Simpson's $1/3^{rd}$ Rule, Simpson's $3/8^{th}$ Rule, Let Us Sum Up

UNIT3 Numerical Solution of Ordinary Differential Equations

Objectives, Introduction, Let Us Sum Up

UNIT 1 NUMERICAL DIFFERENTIATION

Unit Structure

1.0 INTRODUCTION:

A numerical differentiation is the process of determining a function's derivative from a given set of values. By numerical differentiation, it is possible to find the derivative at a given point of a function y = f(x), which cannot be evaluated analytically. One of the interpolation formulas is used to represent the tabulated function, which is then differentiated a certain number of times to provide an approximate value for the derivative. The interpolation formulas of Newton, Stirling, or Bessels apply if the function is tabulated at equal intervals. If the function is tabulated at unequal intervals, then Lagrange's interpolation formula is applicable.

Newton's forward difference interpolation formula is applicable if the derivative of a tabulated function is required at a point near the beginning of the table, while Newton's backward interpolation formula is more appropriate at a point near the table's end. Formulas based on central differences can be used to find the derivative near the center of the table.

Numerical differentiation only provides approximations to derivatives, since round offs and Truncations cannot be avoided. In addition to originating within the method themselves, such errors also increase with each successive differentiation steps.

1.1 DIFFERENTIATING A FUNCTION:

If the function is in form of graphical form., then we can find the derivative of the function at $x = x_0$.

It’s derivative is given by:

$$f'(x_0) = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

On Taking another approximations to the slope by using the line segment of graph

$$f'(x_0) = \frac{f(x_0) - f(x_0 - \delta x)}{\delta x}$$

We can even have a third approximation to the slope using the mean of the first two approximation

$$f'(x_0) = \frac{f(x_0 + \delta x) - f(x_0 - \delta x)}{2\delta x}$$

Using the above approximations we can find the slope of line given in graphical method.

1.2 DIFFERENTIATING A TABULATED FUNCTION:

1.2.1 Function Tabulated at Equal Intervals

It is assumed that function f(x) has n equally spaced points and that we wish to calculate the derivative at the point between x_k and x_{k+1} and in the upper half of the table. In order to calculate the forward differences at a point x, Newton's forward difference interpolating polynomial is presented in the following manner:

$$y(x) = \Delta y_k + \Delta y_k u + \frac{\Delta^2 y_k}{2!} u(u-1) + \frac{\Delta^3 y_k}{3!} u(u-1)(u-2) + ...$$

Where $u = (\frac{x - x_0}{h})$ and $\Delta y_0, \Delta^2 y_0, ...$ are forward difference of the function values at a point $x = x_{0.}$

And on differentiating above equation with respect to u we obtain;

$$\frac{dy}{du} = \Delta y_k + \frac{\Delta^2 y_k}{2!}[(u-1)+u] + \frac{\Delta^3 y_k}{3!}[(u-1)(u-2)+u(u-2)+u(u-1)] + \ldots$$

1.2.2 Function Tabulated at Unequal Intervals

It is assumed that function f(x) has n unequally spaced points and that we wish to calculate the derivative at the point between x_k and x_{k+1} . In order to calculate the forward differences at a point x_k, Newton's divided difference interpolating polynomial is presented in the following manner:

$$y(x) = y_k + \Delta_d y_k (x - x_k) + \Delta_d^2 y_k (x - x_k) + \Delta_d^3 y_k (x - x_k)(x - x_{k+1})(x - x_{k+2}) + \ldots$$

Where $\Delta_d y_k, \Delta_d^2 y_k, \ldots$ are divided differences of the function values at a point $x=x_k$.

On differentiating the above equation with respect to x we obtain;

$$\frac{dy(x)}{dx} = \Delta_d y_k + \Delta_d^2 y_k[(x - x_k) + (x - x_{k+1})] + \Delta_d^3 y_k[(x - x_{k+1})(x - x_{k+2}) + (x - x_k)(x - x_{k+2}) + (x - x_k)(x - x_{k+1})] + \ldots$$

Check Your Progress-1

1. Given the following table

X	0.0	0.25	0.50	0.75	1.0
Y	0.17	0.33	0.59	0.71	1.3

   Find $f^1(0.9)$
2. Using the data of above problem, evaluate $f^1(1.6)$
3. The distance covered by a vehicle in the given time duration is given in the following table:

Time(s)	10	20	30	40	50
Distance(m)	15	30	45	60	75

1.3 LET US SUM UP

In this chapter we learnt many different methods to find the derivative of a function
Some of the methods we learnt in this chapter are :

i. Differentiating a Graphical Function
ii. Differentiating a Tabulated Function

1.4 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. 2.5556
2. 52.3601
3. 1.5

1.5 ASSIGNMENT

1. From the data given in the following table

X	0	1	2	3	4	5
Y	0	3	3.75	4.4	4.2	3.5

Use forward difference formula to compute approximately the value of $\frac{dy}{dx}$ at

i. x = 4.5
ii. x = 6.5

2. Consider the data of problem from above question. Can the difference formula be used to estimate $\frac{dy}{dx}$ at x=7. If not, what could be done if it was required to estimate $\frac{dy}{dx}$.

1.7 ACTIVITY

1. From the Following table, calculate the derivative at x=9

X	0	2	4	6	8
Y	7	23	34	53	70

1.8 FURTHER READING

- M.K. Jain, S.R.K. Iyengar And R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.

- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 2: NUMERICAL INTEGRATION

Unit Structure

2.0 INTRODUCTION:

The process of numerical integration involves computing the value of a definite integral from a set of numerical values of the function called an integrand. It is sometimes called numerical quadrature when applied to the integration of a function of a single variable. Calculating the double integral of a function of two independent variables is known as numerical cubature. Calculus is usually used to determine the integral of a function defined as a mathematical expression.

We will discuss the use of numerical integration techniques to compute integrals of functions of a single variable in this chapter, as well as the errors associated with these methods, and finally, we will develop algorithms for implementing some of these algorithms and leave the rest to the reader. Note that we will consider closed formulae only.

2.1 Newton-Cotes Integration Formulae

In order to obtain Newton-Cotes formulas, we can use the above equation in a variety of ways. Integrating polynomials of varying degrees are represented by these rules. In particular, the first three, with a degree of polynomial equal to 1, 2, or 3, are known as the Trapezoidal rule, Simpson's 1/3rd rule, and Simpson's 3/8th rule.

2.2.1 Trapezoidal Rule

Setting n = 1 in general formula,

$$\int_{x_0}^{x_n} ydx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...]$$

And all the differences higher than the first will become zero and we get

$$I_0 = \int_{x_0}^{x_1} ydx = h[y_0 + \frac{1}{2}\Delta y_0] = h[y_0 + \frac{1}{2}(y_1 - y_0)] = \frac{h}{2}(y_0 + y_1)$$

$$I_1 = \int_{x_1}^{x_2} ydx = \frac{h}{2}(y_1 + y_2)$$

And so on. For the interval [x_n, x_{n+1}], we have

$$I_n = \int_{xn}^{x_{n+1}} ydx = \frac{h}{2}(y_{n+1} + y_n)$$

And, on combining all the above equations, we get

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n]$$

Which is known as trapezoidal rule.

2.2.2 Simpson's 3/1th Rule

This rule can be obtained by placing the value of n as 2 in the below equation

$$\int_{x_0}^{x_n} ydx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...]$$

We then have,

$$I_0 = \int_{x_0}^{x_2} ydx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$I_1 = \int_{x_2}^{x_4} ydx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

And so on. For the interval [x_n, x_{n-2}], we have,

$$\int_{x_{n-2}}^{x_n} ydx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

And on combining all above equations we get

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2}) + y_n]$$

Which is known as Simpson's 1/3 Rule.

2.2.3 Simpson's 3/8th Rule

This rule can be obtained by placing the value of n as 3 in the below equation

$$\int_{x_0}^{x_n} ydx = nh[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + ...]$$

We then have,

$$I_0 = \int_{x_0}^{x_3} ydx = 3h[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0] = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_3}^{x_6} ydx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

On combining the above equation we get

$$\int_{x_0}^{x_n} ydx = \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + ... + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 3y_6 + ... + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Which is known as Simpson's 3/8th Rule.

Check Your Progress-1

1.Find the solution using Trapezodial Rule

x:	0	0.1	0.2	0.3	0.4
y:	1	0.9975	0.99	0.9776	0.8604

2. Find the solution using Simpson's 1/3 Rule

x:	0.00	0.25	0.50	0.75	1.00
y:	1.0000	0.9896	0.9589	0.9089	0.8415

3. Find the solution using Simpson's 3/8 Rule

x:	5	10	15	20	25
y:	100	250	340	400	600

Example 2.1

From the following table, find the area bounded by the curve and x-axis from x=7.47 to x=7.52 using trapezoidal, simplson's 1/3, simplson's 3/8 rule.

x	7.47	7.48	7.49	7.50	7.51	7.52
f(x)	1.93	1.95	1.98	2.01	2.03	2.06

Solution:

The value of table for *x* and *y*

x	7.47	7.48	7.49	7.5	7.51	7.52
y	1.93	1.95	1.98	2.01	2.03	2.06

Using Trapezoidal Rule

$\int y dx = h/2[y_0+y_5+2(y_1+y_2+y_3+y_4)]$

$\int y dx = 0.012[1.93+2.06+2\times(1.95+1.98+2.01+2.03)]$

$\int y dx = 0.012[1.93+2.06+2\times(7.97)]$

$\int y dx = 0.0997$

Solution by Trapezoidal Rule is 0.0997

Example 2.2

Find Solution using Simpson's 1/3 rule

x	f(x)
1.4	0.4055
1.6	0.5876
1.8	0.6842
2.0	0.8641
2.2	0.9264

Solution:

The value of table for *x* and *y*

x	1.4	1.6	1.8	2	2.2
y	0.4055	0.5876	0.6842	0.8641	0.9264

Using Simpsons $1/3^{th}$ Rule

$\int ydx = h/3[(y_0+y_4)+4(y_1+y_3)+2(y_2)]$

$\int ydx = 0.2/3[(0.4055+0.9264)+4\times(0.5876+0.8641)+2\times(0.6842)]$

$\int ydx = 0.2/3[(0.4055+0.9264)+4\times(1.4517)+2\times(0.6842)]$

$\int ydx = 0.5671$

Solution by Simpson's 1/3th Rule is 0.5671

Example 2.3

Find Solution of an equation 2x^3-4x+1 using Simpson's 3/8 rule
x1 = 2 and x2 = 4
Interval N = 5

Solution:
Equation is $f(x)=2x3-4x+1$.

$h=b-a/N$

$h=4-2/5=0.4$

The value of table for x and y

x	2	2.4	2.8	3.2	3.6	4
y	9	19.048	33.704	53.736	79.912	113

Using Simpson's 38 Rule

$\int ydx = 3h/8[(y0+y5)+2(y3)+3(y1+y2+y4)]$

$\int ydx = 3\times0.4/8[(9+113)+2\times(53.736)+3\times(19.048+33.704+79.912)]$

$\int ydx = 3\times0.4/8[(9+113)+2\times(53.736)+3\times(132.664)]$

$\int ydx = 94.1196$

Solution by Simpson's 38 Rule is 94.1196

2.2 LET US SUM UP

We discussed the use of numerical integration techniques to compute integrals of functions of a single variable in this chapter, using three methods

- Trapezoidal Rule
- Simpson’s 3/1th Rule
- Simpson’s 3/8th Rule

2.3 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. 0.3895
2. 0.9445
3. 6700

2.4 GLOSSARY

1. Numerical Integration over more than one dimension is descried as **cubature**.
2. A function which is to be integrated is known as **integrand**.
3. The process of constructing a square with an area equal to that of a circle, or of another figure bounded by a curve is known as **quadrature**.

2.5 ASSIGNMENT

1. From the following table, find the area bounded by the curve and x-axis from x = 7.40 to x = 7.81 using trapezoidal, simpson's 1/3, simpson's 3/8 rule.

x	7.40	7.47	7.56	7.61	7.71	7.81
f(x)	1.93	1.95	1.98	2.01	2.03	2.06

2. Find Solution using Trapezoidal rule

x	f(x)
0.00	1.0000
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415

2.6 ACTIVITY

1. Evaluate $I = \int_0^1 \frac{1}{1+x}$ by using simpson's rulw with h=0.3 and h = 0.6
2. Find Solution using Trapezoidal, Simpson's 1/3, Simpson's 3/8 Rule

x	f(x)
1.4	4.0552
1.6	4.9530
1.8	6.0436
2.0	7.3891

2.2	9.0250

2.7 FURTHER READING

- M.K. Jain, S.R.K. Iyengar And R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.
- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 3: NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

Unit Structure

3.0 INTRODUCTION

In this chapter we will assume the numerical solution of differential equations of type

$$\frac{dy}{dx} = f(x, y)$$

With an initial Approximation as

$$y = y_1 \text{ at } x = x_1$$

It is possible for f(x, y) to be a nonlinear function of (x, y), or it may be a table of values. When the value of y is to be found at $x = x_1$, and the solution is to be found for $x_1 \le x \le x_f$, then the problem is called an initial value problem. If, however, y is given at $x = x_f$, and the solution is required for $x_f >$ x1, then the problem is classified as a boundary value problem. In this book we will consider only the initial value problem.

3.1 EULER'S METHOD

One of the simplest and oldest methods is Euler's method. Euler's method involves developing a piecewise linear approximation to the solution. For the initial value problem, the slope of the solution curve is given as well as the starting point. A solution curve is extrapolated using the specified step size based on this information. Take a look at the following first order ordinary differential equation

$$\frac{dy}{dx} = f(x, y)$$

With the initial conditions as:

$$y = y_1 \text{ for } x = x_1$$

Using the mean value theorem we can find the solution of the above problem

And, the mean value theorem states that,

"If a function is continuous and differentiable between two points $A(x_1,y_1)$ and $B(x_2,y_2)$, then the slope of the line joining these points is equal to the derivative of the function at least at one other point c between these two points"

$$y'(c) = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$

Substituting $c = x_1$ and $h = x_2 – x_1$, in the above equation we get

$$y(x_2) = y(x_1) + hf(x_1, y_1)$$

And, we know that $y'(x_1) = f(x_1, y_1)$

So,

$$\Rightarrow hf(x_1, y_1) = y(x_2) - y(x_1)$$

$$\Rightarrow y(x_2) = y(x_1) + hf(x_1, y_1)$$
$$\Rightarrow y_3 = y_2 + hf(x_2, y_2)$$

Similarly, if we take (x_2,y_2) as the starting point,

$$\Rightarrow y_3 = y_2 + hf(x_2, y_2)$$

And we can write generalized form of the above equations as;

$$\Rightarrow y_{i+1} = y_i + hf(x_i, y_i)$$

Example 3.1

Find y(0.5) for $y'=-2x+3y$, $x_0=0, y_0=-1$, with step length 0.1 using Euler method.

Solution:

Given $y'=-2x+3y, y(0)=-1, h=0.1, y(0.5)=?$

Euler method

$y_1 = y_0 + hf(x_0,y_0) = -1+(0.1)f(0,-1) = -1+(0.1)\cdot(-3) = -1+(-0.3)= -1.3$

$y_2 = y_1+hf(x_1,y_1) = -1.3+(0.1)f(0.1,-1.3) = -1.3+(0.1)\cdot(-4.1) = -1.3+(-0.41)= -1.71$

$y_3 = y_2+hf(x_2,y_2) = -1.71+(0.1)f(0.2,-1.71) = -1.71+(0.1)\cdot(-5.53) = -1.71+(-0.553) = -2.263$

$y_4 = y_3+hf(x_3,y_3)= -2.263+(0.1)f(0.3,-2.263) = -2.263+(0.1)\cdot(-7.389) = -2.263+(-0.7389) = -3.0019$

$y_5 = y_4+hf(x_4,y_4) = -3.0019+(0.1)f(0.4,-3.0019) = -3.0019+(0.1)\cdot(-9.8057) = -3.0019+(-0.9806) = -3.9825$

$\therefore y(0.5) = -3.9825$

Example 3.2

Find y(0.7) for $y'=7x-9y/3$, $x_0=0, y_0=1$, with step length 0.2 using Euler method.

Solution:

Given $y'=7x-9y3, y(0)=1, h=0.2, y(0.7)=?$

Euler method

$y_1 = y_0 + hf(x_0,y_0) = 1+(0.2)f(0,1) = 1+(0.2)\cdot(-3) = 1+(-0.6) =0.4$

$y_2 = y_1+hf(x_1,y_1) = 0.4+(0.2)f(0.2,0.4) = 0.4+(0.2)\cdot(-0.7333) = 0.4+(-0.1467) = 0.2533$

$y_3 = y_2+hf(x_2,y_2) = 0.2533+(0.2)f(0.4,0.2533) = 0.2533+(0.2)\cdot(0.1733) = 0.2533+(0.0347) = 0.288$

$\therefore y(0.6)=0.288$

Check Your Progress-1

1. Find y(0.5) for y’ = -2x-y, y(0) = -1, with step length 0.1

2. Find y(2) for $y' = \frac{x-y}{2}$, y(0)=1, with step length 0.2

3.2 RUNGE-KUTTA SECOND ORDER METHODS

Each of the Runge-Kutta second order methods matches the Taylor series method up to the second degree terms in h, where h is the step size. For these methods, the interval [x_i, x_f] is divided into subintervals and all derivatives (slopes) within those intervals are averaged to determine the dependent variable. In theory, like Euler's method, these methods are one-step methods, in which information at the preceding (x_i, y_i) point is all we need to evaluate y_{i+1}.

Consider the following differential equation

$$\frac{dy}{dx} = f(x, y)$$

With the initial conditions as:

$$y = y_1 \text{ for } x = x_1$$

At the starting point, Let the slope be $s_1 = f(x_1, y_1)$.

Now let the second point be $x_2 = x_1 + h$ where $y_2 = y_1 + s_1h$ and let the slope be $s_2 = f(x_2, y_2)$.

So, the average value of the slope would be $s = (s_1 + s_2)/2$. And then the value of the solution would be calculated using this new average value of slope as

$$y_2 = y_1 + sh$$

Similarly, general form of formula for third, fourth terms. We can write the value of y at the (i+1)th position as

$$y_{i+1} = y_i + sh$$

where, $s = \frac{s_i + s_{i+1}}{2}$

$$s_i = f(x_i, y_i)$$
$$s_{i+1} = f(x_i + h, y_i + s_ih)$$

Example 3.3

Find y(0.7) for $y'=x^2y$, $x_0 = 0$, $y_0 = -1$, with step length 0.1 using Runge-Kutta 2 method

Solution:

Given $y'=x^{2y}$, $y(0) = -1$, $h=0.1$, $y(0.7)=?$

$k_1 = hf(x_0,y_0) = (0.1)f(0,-1) = (0.1)\cdot(0) = 0$
$k_2 = hf(x_0+h,y_0+k_1) = (0.1)f(0.1,-1) = (0.1)\cdot(-0.01) = -0.001$
$y_1=y_0+k_1+k_2/2 = -1-0 = -1.0005$
$\therefore y(0.1) = -1.0005$

Again taking (x_1,y_1) in place of (x_0,y_0) and repeat the process
$k_1 = hf(x_1,y_1) = (0.1)f(0.1,-1.0005) = (0.1)\cdot(-0.01) = -0.001$
$k_2= hf(x_2+h,y_1+k1) = (0.1)f(0.2,-1.0015) = (0.1)\cdot(-0.0401) = -0.004$

$y_2 = y_2+k_2+k_2/2$ = -1.0005-0.0025 = -1.003
$\therefore y(0.2)$ = -1.003

Again taking (x_2,y_2) in place of (x_1,y_1) and repeat the process
$k_1 = hf(x_2,y_2) = (0.1)f(0.2,-1.003) = (0.1)\cdot(-0.0401) = -0.004$
$k_2 = hf(x_2+h,y_2+k_1) = (0.1)f(0.3,-1.007) = (0.1)\cdot(-0.0906) =-0.0091$
$y_3 = y_2+k_1+k_2/2$ = -1.003-0.0065 = -1.0095
$\therefore y(0.3)$ = -1.0095

Again taking (x_3,y_3) in place of (x_2,y_2) and repeat the process
$k_1 = hf(x_3,y_3) = (0.1)f(0.3,-1.0095) = (0.1)\cdot(-0.0909) = -0.0091$
$k_2 = hf(x_3+h,y_3+k_1) = (0.1)f(0.4,-1.0186) = (0.1)\cdot(-0.163) = -0.0163$
$y_4 = y_3+k_1+k_2/2$ = -1.0095-0.0127 = -1.0222
$\therefore y(0.4)$ = -1.0222

Again taking (x_4,y_4) in place of (x_3,y_3) and repeat the process
$k_1 = hf(x_4,y_4) = (0.1)f(0.4,-1.0222) = (0.1)\cdot(-0.1636) = -0.0164$
$k_2 = hf(x_4+h,y_4+k1) = (0.1)f(0.5,-1.0386) = (0.1)\cdot(-0.2596) = -0.026$
$y_5 = y_4+k_1+k_2/2$ = -1.0222-0.0212 = -1.0434
$\therefore y(0.5)$ = -1.0434

Again taking (x_5,y_5) in place of (x_4,y_4) and repeat the process
$k_1 = hf(x_5,y_5) = (0.1)f(0.5,-1.0434) = (0.1)\cdot(-0.2608) = -0.0261$
$k_2 = hf(x_5+h,y_5+k1) = (0.1)f(0.6,-1.0695) = (0.1)\cdot(-0.385) = -0.0385$
$y_6 = y_5+k_1+k_2/2$ = -1.0434-0.0323 = -1.0757
$\therefore y(0.6)$ = -1.0757

Again taking (x_6,y_6) in place of (x_5,y_5) and repeat the process
$k_1 = hf(x_6,y_6) = (0.1)f(0.6,-1.0757) = (0.1)\cdot(-0.3872) = -0.0387$
$k_2 = hf(x_6+h,y_6+k_1) = (0.1)f(0.7,-1.1144) = (0.1)\cdot(-0.5461) = -0.0546$
$y_7 = y_6+k_1+k_2/2$ = -1.0757-0.0467 = -1.1224
$\therefore y(0.7)$ = -1.1224

Example 3.4

Find y(0.9) for $y'=xy$, $x_0=0,y_0=-1$, with step length 0.3 using Runge-Kutta 2 method.
Solution:
Given $y'=xy,y(0)=-1,h=0.3,y(0.9)=?$

$k_1 = hf(x_0,y_0) = (0.3)f(0,-1) = (0.3)\cdot(0) = 0$
$k_2 = hf(x_0+h,y_0+k_1) = (0.3)f(0.3,-1) = (0.3)\cdot(-0.3) = -0.09$
$y_1 = y_0+k_1+k_2/2$ = -1-0.045 = -1.045
$\therefore y(0.3)$ = -1.045

Again taking (x_1,y_1) in place of (x_0,y_0) and repeat the process
$k_1 = hf(x_1,y_1) = (0.3)f(0.3,-1.045) = (0.3)\cdot(-0.3135) = -0.094$
$k_2 = hf(x_1+h,y_1+k_1) = (0.3)f(0.6,-1.139) = (0.3)\cdot(-0.6834) = -0.205$
$y_2 = y_1+k_1+k_2/2$ = -1.045-0.1495 = -1.1945
$\therefore y(0.6)$ = -1.1945

Again taking (x_2,y_2) in place of (x_1,y_1) and repeat the process
$k_1 = hf(x_2,y_2) = (0.3)f(0.6,-1.1945) = (0.3)\cdot(-0.7167) = -0.215$
$k_2 = hf(x_2+h,y_2+k_1) = (0.3)f(0.9,-1.4096) = (0.3)\cdot(-1.2686) = -0.3806$

$y_3 = y_2+k_1+k_2/2$ = -1.1945-0.2978 = -1.4923
$\therefore y(0.9)$ = -1.4923

Check Your Progress-2

1. Find y(0.3) for y' = -(xy^2 + y), y(0) = 1, with step length 0.1

2. Find y(0.2) for y' = -y, y(0) = 1, With step length 0.1

3.3 RUNGE-KUTTA FOURTH ORDER METHODS

Consider the following differential equation

$$\frac{dy}{dx} = f(x, y)$$

With the initial conditions as:

$y = y_1$ for $x = x_1$

Let the starting point be x_1 at which y_1 and let the slope be $s_1 = f(x_1,y_1)$
Let the second point be at $x_2 = x_1 + (h/2)$ at which $y_2 = y_1 +s_1(h/2)$ and let the slope be $s_2 = f(x_2,y_2)$.
Let the third point be at $x_3 = x_1 + (h/2)$ at which $y_3 = y_1 + s_2(h/2)$ and let the slope be $s_3 = f(x_3,y_3)$.
Let the third point be at $x_4 = x_1 + h$ at which $y_4 = y_1 + s_3h$ and let the slope be $s_4 = f(x_4,y_4)$.
Let the average of the slope be given by $s = (s_1 + 2s_2 + 2s_3 + s_4)/6$
Thus the value of the dependent variable y is computed as

$$y_2 = y_1 + hs$$

In general the value of y at (i+1)th point of solution curve is obtained from ith point using the following equation.

$$y_{i+1} = y_i + hs$$

<u>Example 3.5</u>

Find y(0.8) for $y'=x+5y$, $x_0=0,y_0=-1$, with step length 0.2 using Runge-Kutta 4 method
Solution:

Given $y'=x+5y$, $y(0)=-1$, $h=0.2$, $y(0.8)=?$

$k_1 = hf(x_0,y_0)=(0.2)f(0,-1)=(0.2)\cdot(-5)=-1$
$k_2 = hf(x_0+h/2,y_0+k_1/2) = (0.2)f(0.1,-1.5) = (0.2)\cdot(-7.4) = -1.48$
$k_3 = hf(x_0+h/2,y_0+k_2/2) = (0.2)f(0.1,-1.74) = (0.2)\cdot(-8.6) = -1.72$
$k_4 = hf(x_0+h,y_0+k_3) = (0.2)f(0.2,-2.72) = (0.2)\cdot(-13.4) = -2.68$
$y_1 = y_0 + 16(k_1+2k_2+2k_3+k_4)$
$y_1 = -1 + 16[-1+2(-1.48)+2(-1.72)+(-2.68)]$
$y_1=-2.68$
$\therefore y(0.2)=-2.68$

Again taking (x_1,y_1)in place of (x_0,y_0) and repeat the process
$k_1 = hf(x_1,y_1) = (0.2)f(0.2,-2.68) = (0.2)\cdot(-13.2) = -2.64$
$k_2 = hf(x_1+h_2,y_1+k_1/2) = (0.2)f(0.3,-4) = (0.2)\cdot(-19.7) = -3.94$
$k_3 = hf(x_1+h_2,y_1+k_2/2) = (0.2)f(0.3,-4.65) = (0.2)\cdot(-22.95) = -4.59$

$k_4 = hf(x_1+h, y_1+k_3) = (0.2)f(0.4,-7.27) = (0.2)\cdot(-35.95) = -7.19$

$y_2 = y_1 + 16(k_1+2k_2+2k_3+k_4)$
$y_2 = -2.68 + 16[-2.64+2(-3.94)+2(-4.59)+(-7.19)]$
$y_2=-7.1617$
$\therefore y(0.4)=-7.1617$

Again taking (x_2,y_2) in place of (x_0,y_0) and repeat the process
$k_1 = hf(x_2,y_2) = (0.2)f(0.4,-7.1617) = (0.2)\cdot(-35.4083) = -7.0817$
$k_2 = hf(x_2+h_2,y_2+k_1/2) = (0.2)f(0.5,-10.7025) = (0.2)\cdot(-53.0125) = -10.6025$
$k_3 = hf(x_2+h_2,y_2+k_2/2) = (0.2)f(0.5,-12.4629) = (0.2)\cdot(-61.8146) = -12.3629$
$k_4 = hf(x_2+h,y_2+k_3) = (0.2)f(0.6,-19.5246) = (0.2)\cdot(-97.0229) = -19.4046$
$y_3 = y_2 + 16(k_1+2k_2+2k_3+k_4)$
$y_3 = -7.1617 + 16[-7.0817+2(-10.6025)+2(-12.3629)+(-19.4046)]$
$y_3=-19.2312$
$\therefore y(0.6)=-19.2312$

Again taking (x_3,y_3) in place of (x_0,y_0) and repeat the process
$k_1 = hf(x_3,y_3) = (0.2)f(0.6,-19.2312) = (0.2)\cdot(-95.5559) = -19.1112$
$k_2 = hf(x_3+h_2,y_3+k_1/2) = (0.2)f(0.7,-28.7868) = (0.2)\cdot(-143.2339) = -28.6468$
$k_3 = hf(x_3+h_2,y_3+k_2/2) = (0.2)f(0.7,-33.5546) = (0.2)\cdot(-167.0728) = -33.4146$
$k_4 = hf(x_3+h,y_3+k_3) = (0.2)f(0.8,-52.6457) = (0.2)\cdot(-262.4287) = -52.4857$
$y_4 = y_3 + 16(k_1+2k_2+2k_3+k_4)$
$y_4 = -19.2312 + 16[-19.1112+2(-28.6468)+2(-33.4146)+(-52.4857)]$
$y_4 = -51.8511$
$\therefore y(0.8)=-51.8511$

Check Your Progress-3

1. Find y(0.2) for $y'=x^2-y$, $x_0=0, y_0=1$, with step length 0.1 using Runge-Kutta 4 method.
2. Find y(0.2) for $y'=x^2y -1$, $x_0=0, y_0=1$, with step length 0.1 using Runge-Kutta 4 method

3.4 LET US SUM UP

In this chapter we learnt different methods like

- Euler's Method
- Runge-Kutta Second Order Method
- Runge-Kutta Fourth Order Method

to obtain values of the function at any given point.

3.5 SUGGESTED ANSWERS FOR CHECK YOUR PROGRESS

Check Your Progress-1

1. -0.7715
2. 0.9075

Check Your Progress-2

1. 0.7143
2. 0.819

Check Your Progress-3

1. 0.9052
2. 0.9003

3.6 GLOSSARY

1. Extrapolates means to extend by inferring unknown values from trends in the known data.
2. Erected means to put together and set upright.

3.7 ASSIGNMENT

1. Find y(0.2) for y' = $-(x^2y^2 + y)$, y(0) = 1.2, with step length 0.6.
2. Find y(0.4) for y' = $-xy^2$, y(0) = 1, with step length 0.5.

3.8 ACTIVITY

1. Find y(0.3) for y' = $-y^2$, y(0) = 1.9, with step length 0.4

3.9 FURTHER READING

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age international publisher, India, 2007.

- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012

UNIT 4: NUMERICAL SOLUTION OF HIGHER ORDER DIFFERENTIAL EQUATIONS

Unit Structure

4.0 HIGHER ORDER DIFFERENTIAL EQUATION:

A system of first order differential equations can be derived from a system of higher order differential equations. Consider the following second order differential equation as an example.

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

With the initial conditions:

$$y(x_1) = y_1 \text{ and } (\frac{dy}{dx})_{x_1} = y_1^1.$$

Now substitute,

$$z = \frac{dy}{dx}$$

And;

$$\frac{dz}{dx} = \frac{d^2y}{dx^2}$$

Hence the equation $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$ can also be written as the system of first order differential equations;

$$\frac{dy}{dx} = z$$

$$\frac{dy}{dx} = g(x, y, z)$$

With the initial conditions;

$$y(x_1) = y_1$$

$$z(x_1) = y_1^1$$

Example 4.1

Find y(0.1) for y″=1+2xy-x2z, x0=0,y0=1,z0=0, with step length 0.1 using Runge-Kutta 2 method (2nd order derivative)

Solution:
Given $y''=1+2xy-x^2z$, y(0)=1, y′(0)=0, h=0.1, y(0.1)=?

put $\frac{dy}{dx} = z$ and differentiate w.r.t. x, we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

We have system of equations

$$\frac{dy}{dx} = z = f(x, y, z)$$

$$\frac{dz}{dx} = 1 + 2xy - x^2 z = g(x, y, z)$$

Method-1 : Using formula $k_2 = hf(x_0 + h, y_0 + k_1)$

Second order R-K method for second order differential equation

$k_1 = h \cdot f(x_0,y_0,z_0) = (0.1) \cdot f(0,1,0) = (0.1) \cdot (0) = 0$

$l_1 = h \cdot g(x_0,y_0,z_0) = (0.1) \cdot g(0,1,0) = (0.1) \cdot (1) = 0.1$

$k_2 = h \cdot f(x_0+h,y_0+k_1,z_0+l_1) = (0.1) \cdot f(0.1,1,0.1) = (0.1) \cdot (0.1) = 0.01$

$l_2 = h \cdot g(x_0+h,y_0+k_1,z_0+l_1) = (0.1) \cdot g(0.1,1,0.1) = (0.1) \cdot (1.199) = 0.1199$

$$y_1 = y_0 + \frac{k_1 + k_2}{2} = 1 + 0.005 = 1.005$$

$\therefore y(0.1)=1.005$

Check Your Progress – 1

1. Given $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

   with initial conditions y(0)=0, and $\frac{dy}{dx}|_{x=0} = 1$

   Find the solution of the above system of equations in the interval [0,0.4] with interval size h=0.1 using the second order Runge-Kutta method.

2. Find y(0.2) for $y'' = xz^2 - y^2$, $x_0 = 1$, $y_0 = 1$, $z_0 = 1$, with step length 0.2 using Runge-Kutta 2 method (2nd order derivative).

4.1 LET'S SUM UP:

In this chapter we learnt how do we find solution for second order derivatives.
Generally by using second-order derivative.

4.2 Suggested Answer for Check Your Progress:

Check Your Progress-1

1.

i	1	2	3	4	5
X_i	**0.0**	**0.1**	**0.2**	**0.3**	**0.4**
Y_i	**0.0**	**0.09**	**0.163**	**0.221**	**0.267**

2. y(0.2) = 0.98

4.3 GLOSSARY:

1. Numerical Integration over more than one dimension is descried as **cubature**.
2. A function which is to be integrated is known as **integrand**.

4.4 ASSIGNMENT:

1. Find the solution of the following differential equation using the second order Runge-Kutta method

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 2e^x$$

With y(3) = 1 for x = 3.1, 3.2, 3.3, 3.4

2. Find the solution of the following differential equation using the second order Ruge-kutta method

$$3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x^2$$

4.5 ACTIVITY:

1. Find y(0.2) for $y'' = x^2z^2 - y^2, x_0 = 0, y_0 = 0, z_0 = 0$, with step length 0.2 using Runge-Kutta 2 method (2nd order derivative)

4.6 FURTHER READING:

- M.K. Jain, S.R.K. Iyengar and R.K. Jain, Numerical Methods for Scientific and Engineering Computation, 6th Ed., New age International Publisher, India, 2007.

- S.S. Sastry, Introductory Method of Numerical Analysis, 5th Edition, PHI Learning Private Limited, New Delhi, 2012.

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